Lecture 3 Multi-variate Regression Model

The model now is:

$$y_t = b_0 + b_1 x_{1t} + \cdots + b_k x_{kt} + u_t \quad t = 1, \ldots, T$$

Assumptions

1. $E(u_t) = 0$
2. $E(u_t^2) = \sigma_u^2$
3. $E(u_t u_s) = 0$ for $t \neq s$
4. $E(u_t x_{it}) = 0$ for all $i = 1, \ldots, k$
5. No linear combinations among regressors.

Assumption (5) is new.

On Ass(5), suppose that $x_{kt} = a_0 + a_1 x_{1t}$. Then model is

$$y_t = (b_0 + a_0 b_k) + (b_1 + a_1 b_k) x_{1t} + \cdots + b_{k-1} x_{k-1,t} + u_t$$

$$\equiv b_0' + b_1' x_{1t} + b_2 x_{2t} + \cdots + b_{k-1} x_{k-1,t} + u_t$$

Then we cannot separately identify b_1 and b_k, ie of x_1 and x_k. This is called perfect collinearity between regressors.

LS estimation;

LS objective:

$$\underset{\underline{b}}{\text{minimize}} \quad \sum_t (y_t - \hat{b}_0 - \hat{b}_1 x_{1t} - \cdots - \hat{b}_k x_{kt})^2$$

Necessary conditions:

$$\frac{\partial}{\partial \hat{b}_0} = -2\sum_t (y_t - \hat{b}_0 - \hat{b}_1 x_{1t} \cdots - \hat{b}_k x_{kt}) = 0$$

$$\frac{\partial}{\partial \hat{b}_i} = -2\sum_t x_{it} (y_t - \cdots - \hat{b}_k x_{kt}) = 0 \qquad i = 1, \ldots, k$$

$$\Rightarrow (1) \quad \bar{y} = \hat{b}_0 + \hat{b}_1 \bar{x}_1 + \cdots + \hat{b}_k \bar{x}_k$$

$$(2) \quad \sum_t x_{it} y_t = \hat{b}_0 \sum_t x_{it} + \hat{b}_1 \sum_t x_{it}^2 + \hat{b}_2 \sum_t x_{it} x_{2t} + \cdots + \hat{b}_k \sum_t x_{it} x_{kt}$$

for $i = 1, \ldots, k$

Note we have $(k+1)$ parameters and $(k+1)$ normal equations. We solve them for $\hat{b}_0, \hat{b}_1, \ldots \hat{b}_k$

<u>Obtain the LS Estimators</u>

Consider k^{th} eq

$$(3) \quad \sum_t x_{kt} y_t = \hat{b}_0 \sum_t x_{kt} + \hat{b}_1 \sum_t x_{1t} x_{kt} + \cdots + \hat{b}_k \sum_t x_{kt}^2$$

Consider the situation where x_{Kt} is regressed against all other regressors. This would yield:

(4) $$X_{Kt} = \hat{c}_0 + \hat{c}_1 X_{1t} + \cdots + \hat{c}_{K-1} X_{K-1,t} + \hat{v}_{Kt}$$
$$\equiv \hat{X}_{Kt} + \hat{v}_{Kt}$$

decomposes X_K into part related to other X's and a part orthogonal to X's. Substituting into normal eq

(5) $$\sum y_t x_{Kt} =$$
$$\sum y_t \hat{x}_{Kt} + \sum y_t \hat{v}_{Kt} = \hat{b}_0 \sum \hat{x}_{Kt} + \hat{b}_1 \sum x_{1t} \hat{x}_{Kt} + \cdots + \hat{b}_K \sum \hat{x}_{Kt}^2$$
$$+ \hat{b}_K \sum \hat{v}_{Kt}^2$$

where we used the fact that:

$$\sum \hat{v}_{Kt} = \sum \hat{v}_{Kt} \hat{x}_{Kt} = 0$$

But, from LS estimation:

$$y_t = \hat{b}_0 + \hat{b}_1 x_{1t} + \cdots + \hat{b}_K x_{Kt} + \hat{u}_t \text{ , so that:}$$

(6) $$\sum y_t \hat{x}_{Kt} = \hat{b}_0 \sum \hat{x}_{Kt} + \hat{b}_1 \sum x_{1t} \hat{x}_{Kt} + \cdots + \hat{b}_K \sum x_{Kt} \hat{x}_{Kt} + \sum \hat{u}_t \hat{x}_{Kt}$$

and we note from (4) that in (6):

$$\sum \hat{u}_t \hat{x}_{Kt} = \hat{c}_0 \sum \hat{u}_t + \hat{c}_1 \sum \hat{u} x_{1t} + \cdots + \hat{c}_{K-1} \sum \hat{u} x_{K-1,t} = 0$$

Using (6) in (5)

Therefore we get from the normal eq.

$$\Sigma y_t \hat{v}_{kt} = \hat{b}_k \Sigma \hat{v}_{kt}^2$$

whence:

$$\hat{b}_k = \frac{\Sigma y_t \hat{v}_{kt}}{\Sigma \hat{v}_{kt}^2}$$ and similarly for $\hat{b}_i$ $i=1,\dots,k-1$

That is, the estimator $\hat{b}_k$ is the ratio of the cov between y and that part of x_k orthogonal to other x's to the variance of the orthogonal part of x_k.

Note: If x_k linearly dependent on other x's, $\hat{v}_k = 0$.
Then $\hat{b}_k$ is not defined separately.

Properties of LS estimators:

Unbiasedness

Consider $\hat{b}_i = \Sigma y_t \hat{v}_{it} / \Sigma \hat{v}_{it}^2$. Write:

$$y_t = b_0 + b_1 x_{1t} + \dots + b_i(\hat{x}_{it} + \hat{v}_{it}) + \dots + b_k x_{kt} + u_t$$

Then $\sum y_t \hat{v}_{it} = b_i \sum \hat{v}_{it}^2 + \sum \hat{v}_{it} u_t$. so

$$\hat{b}_i = b_i + \sum \alpha_i u_t \qquad \alpha_i = \hat{v}_{it} / \sum \hat{v}_{it}^2$$

$$E(\hat{b}_i) = b_i$$

qed

<u>Variance</u>

$$\sigma^2_{\hat{b}_i} = \sum \alpha_i^2 \, E(u_t^2) = \sigma_u^2 \sum \alpha_i^2$$

Since $\sum \alpha_i^2 = \dfrac{\sum \hat{v}_{it}^2}{\left(\sum \hat{v}_{it}^2\right)^2} = \dfrac{1}{\sum \hat{v}_{it}^2}$

$$\therefore \quad \sigma^2_{\hat{b}_i} = \frac{\sigma_u^2}{\sum \hat{v}_{it}^2}$$

We estimate σ_u^2 as

$$\hat{\sigma}_u^2 = \frac{\sum_t \hat{u}_t^2}{T-(k+1)} \qquad \hat{u}_t = y_t - \hat{b}_0 - \hat{b}_1 x_{1t} - \dots - \hat{b}_k x_{kt}$$

The explanatory power of the regression is measured exactly as in the bivariate case. We have

$$TSS = ESS + RSS$$

i.e. $\Sigma(y_t - \bar{y})^2 = \Sigma(\hat{y}_t - \bar{y})^2 + \Sigma \hat{u}_t^2$

where $\hat{u}_t = y_t - \hat{y}_t$.

$$R^2 = 1 - \frac{RSS}{TSS} = \frac{ESS}{TSS}$$

On this basis we can test the hypothesis of no explanatory power formally

Tests of hypotheses

Simple hypothesis: $H_0: b_i = b_i^\circ$

$$\frac{\hat{b}_i - b_i^\circ}{\sigma_{\hat{b}_i}} \sim N(0,1) \qquad \hat{b}_i \pm 1.96\,\sigma_{\hat{b}_i} \quad 95\%\ \text{C.I.}$$

$$\frac{\hat{b}_i - b_i^\circ}{\hat{\sigma}_{\hat{b}_i}} \sim t_{N-K-1}$$

K = # regressors, exc. constant

Example

Life-Cycle Consumption (Ando & Modigliani)

Leads to estimated equation

$$C_t = b_0 + b_1 y_{dt} + b_2 A_{t-1} + U_t$$

y_{dt} = current disposable labor income, A_{t-1} = last period assets (net worth) as measure of expected nonlabor income.

Estimation:

$$\hat{C}_t = \underset{(1.46)}{5.33} + \underset{(.047)}{.767}\, y_{dt} + \underset{(.010)}{.047}\, A_{t-1} \qquad T = 25,\ R^2 = .999$$

Note: est. std errors in parentheses.

Since $\hat{b}_i / \hat{\sigma}_{b_i} > 2 \quad \forall i$, we reject H_0 that $b_0 = 0$, $b_1 = 0$ or $b_2 = 0$.

Other hypothesis:

$H_0: \ b_1 = 1$

$H_1: \ b_1 \neq 1$

$$TS = \frac{.767 - 1}{.047} = \frac{.233}{.047} \doteq 5$$

Reject H_0.

Suppose we also wanted to include a variable, $Z = r A_{t-1}$, interest income.

$$C_t = b_0 + b_1 y_{dt} + b_2 A_{t-1} + b_3 Z_{t-1} + U_t$$

If $r = \bar{r}$, the true model is equivalent to

$$C_t = b_0 + b_1 y_{dt} + (b_2 + b_3 \bar{r}) A_{t-1} + U_t$$

and collinearity does not permit identification of b_2 and b_3 separately. Does this hold if r varies over time?