Remarks on Functional Form : Linearity in Parameters

Linear models are models which can be put into a form which is linear in the parameters to be estimated. Nonlinear models are not linear in parameters.

Examples of Transformations

1. Reciprocal Transform / Phillips Curve

Adjustment eq. in wages

$$\dot{w}_t \equiv \frac{W_t - W_{t-1}}{W_{t-1}} = a D_t^* + u_t$$

$D_t^* = (D_t - S_t)/S_t$ rate of excess demand.

$D_t^* = b + c\,(1/X)_t$ $\quad X_t$ = unemployment rate

As $X_t \to \infty$ $\quad D_t^* \to b$.

Substituting,

$$\dot{w}_t = ab + ac\,(1/X_t) + u_t$$

Define $Z = 1/X$. Then we have linear relation:

$$\dot{w}_t = \alpha + \beta Z_t + u_t$$

2. Log Transform : Cobb Douglas P.F. (Log Linear)

$$Q_t = aL^b C^c e^{U_t}$$

$$\log Q_t = \log a + b \log L + c \log C + U_t$$

Define $y = \log Q$, $x = \log L$, $z = \log C$. Then

$$y_t = a' + b x_t + c z_t + U_t$$

3. Semi-Log Transform : Growth Rates

$$P = A e^{gt} e^{U_t}$$

$$\log P = \log A + gt + U_t$$

Define $y_t = \log P_t$. Then

$$y_t = a + gt + U_t$$

Scaling of Variables

Suppose

$$y_t = a + b x_t + u_t$$

and $Y_t = \lambda_1 y_t$, $X_t = \lambda_2 x_t$. Then in observed variables:

$$Y_t = \lambda_1 a + \left(\frac{\lambda_1}{\lambda_2}\right) b X_t + \lambda_1 u_t$$

$$= A + B X_t + U_t$$

Then $\hat{A} = \lambda_1 \hat{a}$, $\hat{B} = (\lambda_1/\lambda_2)\hat{b}$ and

$$\hat{\sigma}^2_{\hat{A}} = \lambda_1^2 \hat{\sigma}^2_{\hat{a}}, \quad \hat{\sigma}^2_{\hat{B}} = \left(\frac{\lambda_1}{\lambda_2}\right)^2 \hat{\sigma}^2_{\hat{b}}.$$

Then $t_{\hat{A}} = \dfrac{\lambda_1 \hat{a}}{\lambda_1 \hat{\sigma}_{\hat{a}}} = \dfrac{\hat{a}}{\hat{\sigma}_{\hat{a}}}$ and $t_{\hat{B}} = \dfrac{(\lambda_1/\lambda_2)\hat{b}}{(\lambda_1/\lambda_2)\hat{\sigma}_{\hat{b}}} = \dfrac{\hat{b}}{\hat{\sigma}_{\hat{b}}}$

Hence scaling does not alter estimated t ratios.

Note: scaling here is nonstochastic.

Hypothesis Tests Involving Several Parameters

Given the model

$$y_t = b_0 + b_1 x_{1t} + \ldots + b_k x_{kt} + u_t$$

Two types of Examples

1. Linear restrictions among parameters

   e.g. $b_1 = -3 b_5$

   $b_1 + b_2 + \ldots + b_k = 1$

2. Joint test of significance of regressors

   e.g.

   $$b_0 = b_1 = \ldots = b_k = 0$$

H_0 : Restrictions hold

H_1 : Restrictions do not hold

Procedure

1. Impose H_0 and estimate the restricted form; obtain the error sum of squares, RSS_r.

2. Estimate unrestricted form, w/out H_0 imposed, and obtain RSS_U.

3. Compute number of restrictions in H_0, d, which is the difference in the number of estimated parameters

in unrestricted and restricted forms.

4. Compute

$$TS = \frac{(RSS_r - RSS_u)/d}{RSS_u / T-K-1}$$

Note that

$RSS_r = \Sigma \hat{U}_{tr}^2$ and $RSS_u = \Sigma \hat{U}_{tu}^2$. Under H_0, $\hat{U}_r \sim N(\cdot)$ and $\hat{U}_u \sim N(\cdot)$. Hence $\hat{U}_r^2 \sim \chi^2$, $\hat{U}_u \sim \chi^2$. Hance RSS_r and RSS_u also distributed as χ^2 variates, which can be shown to be independent of each other. Therefore, $TS \sim F_{d, T-K-1}$ since it's the ratio of two independent χ^2 divided by their degrees of freedom.

Why does $\hat{U}_r \sim N(\cdot)$? Under H_0, both the restricted and unrestricted models are properly specified eqs; and hence RSS_r and RSS_u differ only by sampling error under H_0. Note of cource $RSS_r \geq RSS_u$.

Conclusion $TS = \frac{(RSS_r - RSS_u)/d}{RSS_u / T-K-1} \sim F_{d, T-K-1}$.

Example : Life Cycle Model of consumption

Test is:

$H_0 : b_1 = b_2$

$H_1 : b_1 \neq b_2$

Estimate

Unrest. 1. $c_t = b_0 + b_1 y_d + b_2 A_{t-1} + u_t$ $T = 25$

Restr. 2 $c_t = b_0 + b(y_d + A_{t-1}) + v_t$ $T = 25$

$\equiv b_0 + b Z_t + v_t \quad ; Z_t = y_{dt} + A_{t-1}$

$$TS = \frac{\sum_t \hat{v}_t^2 - \sum_t \hat{u}_t^2}{\sum_t \hat{u}_t^2} \cdot \frac{(25-2-1)}{1} \gtrless F_{1,23} \simeq 4.2$$

$F_{1,22} = 4.30$

Functional Form: Multivariate Regression

Consider Cobb Douglas production function with capital and labor

$$Q_t = A L_t^{b_1} K_t^{b_2} e^{u_t}$$

Taking logs:

$$q_t = a + b_1 l_t + b_2 k_t + u_t$$

small letters denote logs, e.g. $l_t = \log L_t$. Consider doubling all inputs. What happens to output?

$$Q'_t = A (2L_t)^{b_1} (2K_t)^{b_2} e^{u_t}$$

$$= A\, 2^{b_1+b_2} L_t^{b_1} K_t^{b_2} e^{u_t}$$

$$Q'_t = 2^{b_1+b_2} Q_t$$

Hence, if $b_1+b_2 = 1$, $Q'_t = 2Q_t$. If $b_1+b_2 \gtrless 1$, $Q'_t \gtrless 2Q_t$

We call $b_1+b_2 = 1$ case of CRS; $b_1+b_2 < 1$ as DRS; $b_1+b_2 > 1$ IRS.

Test of CRS (constant returns to scale)

H_0: $b_1+b_2 = 1$

H_1: $b_1+b_2 \neq 1$

1. Estimate

$$q_t = a + b_1 l_t + b_2 k_t + u_t$$

get $\sum \hat{u}_t^2$

2. Estimate under Ho,

$$q_t = a + b_1 l_t + (1-b_1) k_t + v_t$$

$$\Rightarrow \quad q_t - k_t = a + b_1 (l_t - k_t) + v_t$$

get $\sum \hat{v}_t^2$.

Compute:

$$TS = \frac{\sum \hat{v}_t^2 - \sum \hat{u}_t^2}{\sum \hat{u}_t^2} \cdot \frac{T-3}{1} \sim F_{1,T-3}.$$

Polynomial Approximations

For unknown nonlinear relationships, we often use a polynomial function of some degree to approximate:

$$y_t = f(x_t) + u_t$$

$$\simeq a_0 + a_1 x_t + a_2 x_t^2 + \dots + a_k x_t^k + u_t$$

which is convertible into a linear regressions w/ k+1 parame
To test that $f(x_t) = 0$ we use the joint significance test

H_0: $a_0 = a_1 = \dots = a_k = 0$
H_1: At least one $a_i \neq 0$.

Note: Whatever approximation error is, it will become part of the "disturbance." If it's large we will generally obtain biased coefficients because error will be correlated with regressors.

Dummy Variables : Analysis of Variance / Covariance

1. Consider consumption function

$$C_t = b_0 + b_1 y_t + u_t \qquad \text{peacetime}$$

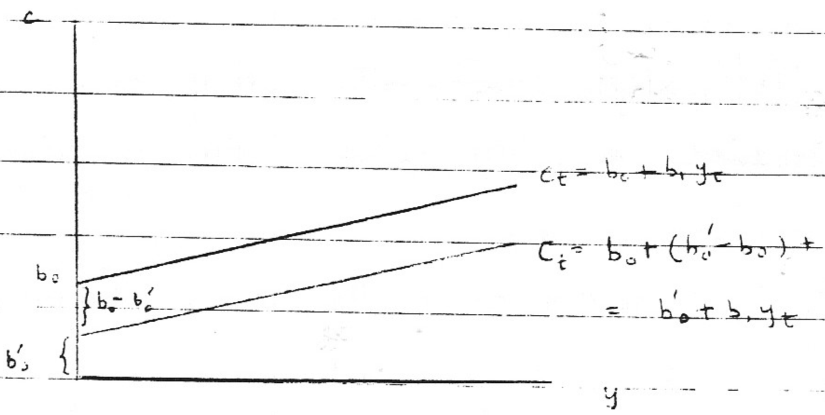
$$C_t = b_0' + b_1 y_t + u_t \qquad \text{wartime}$$

$$= b_0 + (b_0' - b_0) + b_1 y_t + u_t$$

We combine them as

$$C_t = b_0 + \gamma D + b_1 y_t + u_t \quad ; \quad D = \begin{cases} 1 & \text{wartime} \\ 0 & \text{peacetime} \end{cases}$$

$$\gamma = b_0' - b_0 .$$



Note : Dummy variable takes on values (0, 1). Intercept variable takes on value 1 always. Hence, no linear dependence. However, consider running the following:

$$C_t = b_0 + b_1 y_t + \gamma_1 D_1 + \gamma_2 D_2 + u_t$$

$$D_1 = \begin{cases} 1 & \text{wartime} \\ 0 & \text{peacetime} \end{cases} \qquad D_2 = \begin{cases} 1 & \text{peacetime} \\ 0 & \text{wartime} \end{cases}$$

Now note D_1 and D_2 are not linearly dependent. But intercept variable $\equiv 1 = D_1 + D_2$. Hence we conclude, if we want two intercepts in the model, we only use one dummy variable plus a constant. In general, for N intercept values, use N-1 dummy variables and an intercept. The dummy coefficient gives intercept relative to b_0.

2. To let slope coefficients vary across regimes, introduce an interactive dummy variable:

$$C_t = b_0 + b_1 y_t + u_t \qquad \text{peace} \sim\sim\text{wartime}\sim\sim$$

$$C_t = b_0 + b_1' y_t + u_t \qquad \text{war} \sim\sim\text{peace}\sim\sim\text{time}$$

$$\Rightarrow \quad C_t = b_0 + b_1 y_t + \gamma (D \cdot y_t) + u_t \quad , \quad \gamma = b_1' - b_1 .$$

Again, for N regimes we use only N-1 dummy variables. The dummy coefficient gives slope relative to b_1.

Dummy variables are an easy way of introducing qualitative factors into a regression. For example, race, ranges of age, ~~etc~~ sex.

- Example from Kelejian & Oates

$$G = 96 - 1.21\,P - .004\,Y - .6\,Z - 15.9\,F \qquad N=53$$
$$(12.1) \quad (1.3) \quad (2.3) \quad (5.5) \quad (4.7) \qquad R^2 = .65$$

where G = % central gov't share revenues
P = log population
Y = per capita income
Z = social security payments as % revenue
$F = \begin{cases} 1 & \text{federal constitution} \\ 0 & \text{nonfederal constitution} \end{cases}$

Hence, F is significant variable, and of economic importance. Countries w/ federal constitutions have 16% points lower central gov't share in revenues than those w/out such constitutions.