Problem of Autocorrelation

$$y_t = b_0 + b_1 x_t + U_t$$

$E(U_t U_s) \neq 0$ serial or autocorrelation in disturbances.
$t \neq s$

Conclusion: We show that usually $E(U) = 0$ and $E(UX) = 0$. Hence estimators remain unbiased. However, formulas for $\sigma^2_{\hat{b}}$ change; when there is autocorrelation we understate the true $\sigma^2_{\hat{b}}$, hence overstate t-ratios.

First order Autoregressive Model

$$U_t = \gamma U_{t-1} + \varepsilon_t \qquad \varepsilon_t \sim N(0, \sigma^2_\varepsilon)$$

$$E(\varepsilon_t \varepsilon_s) = 0 \qquad t \neq s$$

$$E(x_t \varepsilon_t) = 0$$

By successive substitutions

$$U_t = \gamma(\gamma U_{t-2} + \varepsilon_{t-1}) + \varepsilon_t$$

$$= \gamma^2 (U_{t-3} + \varepsilon_{t-2}) + \gamma \varepsilon_{t-1} + \varepsilon_t$$

$$= \dots = \varepsilon_t + \gamma \varepsilon_{t-1} + \gamma^2 \varepsilon_{t-2} + \dots$$

$$U_t = \sum_{i=0}^{\infty} \gamma^i \varepsilon_{t-i}$$

Note:

$$E(U_t) = \sum_{i=0}^{\infty} \gamma^i E(\varepsilon_{t-i}) = 0$$

$$E(x_t U_t) = \sum_{i=0}^{\infty} \gamma^i E(x_t \varepsilon_{t-i}) = 0$$

$$\sigma_{U_t}^2 = E(U_t^2) = E\left[\sum_{i=0}^{\infty} \gamma^{2i} \varepsilon_{t-i}^2 + \text{cross terms}\right]$$

$$= \sigma_\varepsilon^2 \sum_{i=0}^{\infty} \gamma^{2i}$$

We note the formula for geometric series,

$$\sum_{i=0}^{\infty} a^i = \frac{1}{1-a} \quad \text{if } |a| < 1$$

In our case $a = \gamma^2$. Hence we write:

$$\sigma_U^2 = \frac{\sigma_\varepsilon^2}{1-\gamma^2} \quad \text{provided } |\gamma| < 1.$$

Correlation pattern is exhibited by autocovariances:

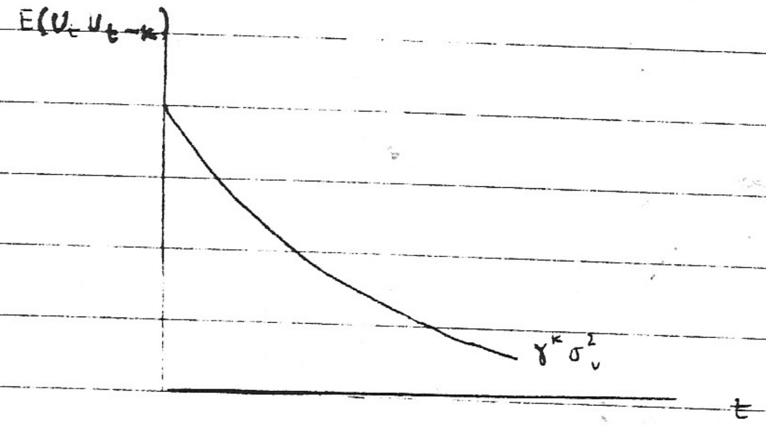
$$E(u_t u_{t-1}) = E(\gamma u_{t-1} + \varepsilon_t) u_{t-1} = \gamma \sigma_u^2$$

$$E(u_t u_{t-2}) = E(\gamma^2 u_{t-2} + \gamma \varepsilon_{t-1} + \varepsilon_t) u_{t-2} = \gamma^2 \sigma_u^2$$

$$\vdots$$

$$E(u_t u_{t-k}) = \gamma^k \sigma_u^2$$

Graph autocovariance function $E(u_t u_{t-k})$.



Hence, for first order AR process autocovariances dampen exponentially.

Consequences of Autocorrelation

1. We derived LS estimators:

$$\hat{b}_1 = b_1 + \sum w_t u_t \quad ; \quad w_t = \frac{x_t - \bar{x}}{\sum (x_t - \bar{x})^2}$$

so

$$E(\hat{b}_1 - b_1)^2 = E[\sum w_t u_t]$$

but now the cross terms do <u>not</u> vanish. Hence our earlier formulas for variances of estimators are not vali

2. $E(\hat{b}_1) = b$ requires only $E(u_t) = 0$. Hence, autocorrelation does not disturb unbiasedness property.

Generalized Least Squares Estimation : Autcorrelation

Model is given by:

$$y_t = b_0 + b_1 x_t + u_t$$

$$u_t = \gamma u_{t-1} + \varepsilon_t \qquad |\gamma| < 1.$$

i) <u>Assume γ is known</u>

Multiply model by γ and lag one period. Subtracting:

$$y_t - \gamma y_{t-1} = (b_0 - \gamma b_0) + b_1 (x_t - \gamma x_{t-1}) + (U_t - \gamma U_{t-1})$$

But $U_t - \gamma U_{t-1} = \varepsilon_t$. Define $y'_t = y_t - \gamma y_{t-1}$ etc. The transformed model is:

$$y'_t = b'_0 + b_1 x'_t + \varepsilon_t$$

where $b'_0 = b_0(1-\gamma)$ and the error is now serially uncorrel
Assumptions are satisfied by the transformed model and hence we use standard formulas. Obtain $\hat{b}'_0$ and $\hat{b}_1$. Retrieve $\hat{b}_0 = \hat{b}'_0 / (1-\gamma)$ and $\sigma^2_{\hat{b}_0} = \frac{1}{(1-\gamma)^2} \sigma^2_{\hat{b}'_0}$.

Note we lose ~~one~~ observation in implementing transformed mod

ii) <u>Unknown γ</u>.

When γ is unknown we can estimate it from the residuals obtained by estimating the original model by least squares. Since $\hat{b}_0$ and $\hat{b}_1$ are still unbiased, so will be $\hat{U}_t$. Hence we can obtain consistent (biased because of dropped observation) estimate of γ

Procedure

Estimate:

$$y = b_0 + b_1 x_t + u_t$$

Obtain:

$$\hat{u}_t = y_t - \hat{b}_0 - \hat{b}_1 x_t$$

Run the regression:

$$\hat{u}_t = \gamma \hat{u}_{t-1} + \varepsilon_t$$

and obtain the LS estimator of γ, $\hat{\gamma}$:

$$\hat{\gamma} = \frac{\sum_{t=2}^{T} \hat{u}_{t-1} \hat{u}_t}{\sum_{t=2}^{T} \hat{u}_{t-1}^2}$$

Use this estimator for γ to transform the model and obtain (asymptotically) efficient estimators of all parameters. Write:

$$y_t^* = b_0^* + b_1 x_t^* + \varepsilon_t$$

where $y_t^* = y_t - \hat{\gamma} y_{t-1}$ etc. Then we obtain

$$\hat{b}_1 = \frac{\sum_{t=2}^{T} (x_t^* - \bar{x}^*)(y_t^* - \bar{y}^*)}{\sum_{t=2}^{T} (x_t^* - \bar{x})^2}$$

$$\hat{b}_0 = \hat{b}_0^* / (1 - \hat{\gamma})$$

and standard variance formulas hold with "starred" variable replacing untransformed (original) ones.

Special Case : First-Differencing

Suppose the error structure is given by

$$v_t = v_{t-1} + \varepsilon_t$$

which is called a random walk. Note that $\sigma_v^2 \to \infty$. The transformed model corresponding to this case is

$$y_t - y_{t-1} = (b_0 - b_0) + b_1 (x_t - x_{t-1}) + \varepsilon_t$$

$$\Delta y_t = b_1 \Delta x_t + \varepsilon_t$$

Note the constant should not appear in the model after -

first-differencing. But this procedure is not recommended in general for autocorrelation; only if $\gamma = 1$. Otherwise it yields biased and inconsistent estimators

General Procedure for Unknown γ

1. Estimate using OLS, and obtain $\hat{u}_t$.
2. Estimate γ from $\hat{u}_t = \gamma \hat{u}_{t-1} + \xi_t$ (no constant).
3. Transform variables as $y_t - \hat{\gamma} y_{t-1}$, $x_t - \hat{\gamma} x_{t-1}$
4. Run OLS on transformed model.

Durbin Watson Test for Autocorrelation

Test is based on sum of squared differences in successive values of residuals:

$$d = \frac{\sum_{t=2}^{T} (\hat{u}_t - \hat{u}_{t-1})^2}{\sum_{t=1}^{T} \hat{u}_t^2}$$

When U_t's are positively related, expect small d. When U's negatively correlated, expect large d. What are the appropriate confidence limits?

$$d = \frac{\sum (\hat{U}_t - \hat{U}_{t-1})^2}{\sum_{t=1}^{T} \hat{U}_t^2} \simeq \frac{2\sum_{t=2}^{T} \hat{U}_t^2 - 2\sum_{t=2}^{T} \hat{U}_t \hat{U}_{t-1}}{\sum_{t=1}^{T} \hat{U}_t^2}$$

where we assume $\sum \hat{U}_{t-1}^2 \simeq \sum \hat{U}_t^2$. Divide by T; assume T large.

$$d \simeq \frac{2\sum \hat{U}_t^2 / T - 2\sum \hat{U}_t \hat{U}_{t-1} / T}{\sum \hat{U}_t^2 / T}$$

$$= \frac{2[\sigma_U^2 - \sigma_{UU_{-1}}]}{\sigma_U^2}$$

$$= 2(1 - \mathrm{cov}(U, U_{-1}) / \sigma_U^2)$$

Since we postulate $U_t = \gamma U_{t-1} + \xi$, we write $\gamma = \mathrm{cov}(U_t U_{t-1}) / \mathrm{var}(U_{t-1}) = \mathrm{cov}(\ ) / \mathrm{var}(U)$. Hence we conclude:

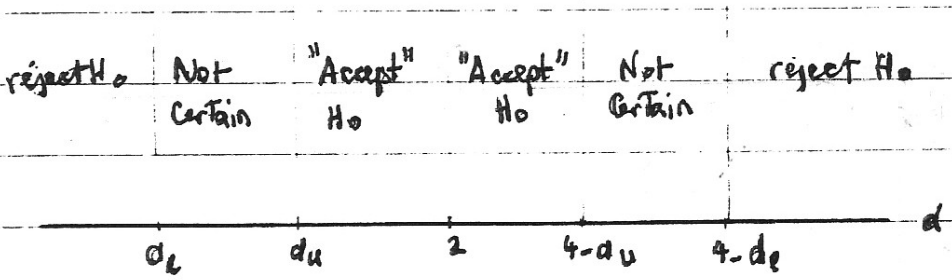
$$d \simeq 2(1-\gamma)$$

To test:

$$H_0 : \gamma = 0$$

$$H_1 : \gamma \neq 0$$

Accept H_0 if $d \simeq 2$, reject otherwise. However, for statistical reasons beyond our concern there are regions of indeterminacy. Specifically,



where d_U and d_L are upper and lower Durbin Watson bounds, which depend on significance level and d.f.

Comments

1. Note d test applies only to AR(1) process.
2. If computed d falls in indeterminate range, do <u>not</u> conclude the absence of autocorrelation.