INTEREST RATE REGIMES

Luigi Vena
02/22/2015
Liuc – Carlo Cattaneo

Today's Agenda

- Course structure
- Finance Dictionary
- Simple Rate
- Compound Rate
- Continuous Rate
- Future Value
- Present Value

International Financial Markets – 02/22/2016

Mishkin, Eakins – ch. 3-4

Instructors

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Student's Assessment

Students attending the course (at least 75% of the classes):

- Home assignments (25%)
- Class presentations (50%)
- Written exam (25%)

Students not attending the course:

- Written exam.

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Interest and interest rates

- Interest: amount of money charged by a lender to a borrower for the use of assets;
- Interest rate: is the interest expressed as percentage of the principal.

Example:

- Principal = 100$ and Interest = 10$ -> Interest rate = 10$/100$ = 10%
- Principal = 100$ and Interest rate = 15% -> Interest = 100$*15% = 15$

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Interest rates

- The 1-year interest rate represents the price paid (as percentage of the principal) for borrowing money in a year.
- Interest rate can be computed at any frequency, not just yearly.
- Interest rate is simply the cost of borrowing or the price paid for the rental of fund.

Finance Dictionary

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I principle of finance: *a dollar today is worth more than a dollar tomorrow.*

- Money can be invested to earn interest.
- Between $100 now and $100 next year, one takes the money now to get a year's interest.

Future Value vs Present Value

- **Future Value:** The value of cash at a specified date in the future that is equivalent in value to a specified sum today.
- **Present Value:** the value that should be assigned now, in the present, to money that is to be received at a later time.

Finance Dictionary

Future Value vs Present Value

- Money received in the future is worth less than the same amount of money received in the present.
- Money received today can be invested to earn interest.
- **Present value** is the discounted magnitude of a cash flow available at a future date.
- **Future value** is the capitalized magnitude of a cash flow available in the present.

Finance Dictionary

From now on, we use the following notation:

P, to indicate the principal i.e.:

- The face value of a bond;
- The amount borrowed or the amount still owed on a loan;
- The original amount invested.

r, to indicate the interest rate;

I, to indicate the interest;

n, total number of periods.

t, the time (usually expressed in years)

Finance Dictionary

Cash Flow and Cash Flow Stream

- Cash flows are the amounts of money that will flow to and from an investor over time.
- Cash flows (either positive or negative) occur at a known specific dates, such as at the end of each month/quarter/year.
- The stream of cash flow can be described by listing flows at each of the date in which they occur.

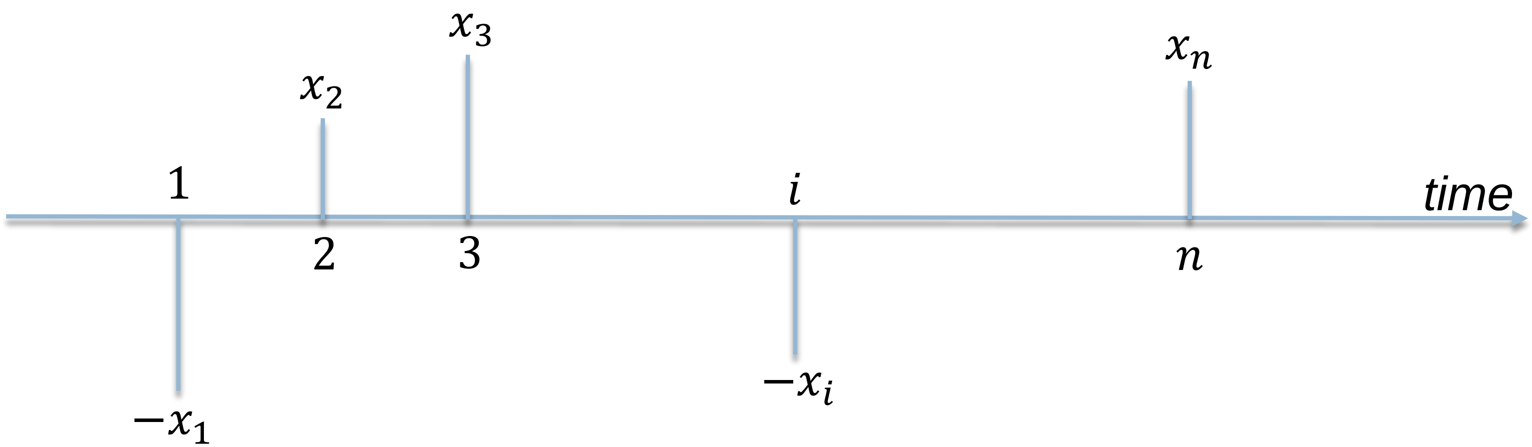
Cash Flow and Cash Flow Stream

- Among others, cash flow stream can be represented by a diagram, where:
- Negative cash flows represent cash outlays.
- Positive cash flows represent cash collections/proceeds.

Finance Dictionary

Cash Flow and Cash Flow Stream Representation

- $(-x_1, x_2, x_3, \dots, -x_i, \dots, x_n | t_1, t_2, t_3, \dots, t_i, \dots, t_n)$



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Simple Rate

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- According to the Simple Rate, Interest are only computed with respect to the principal.
- It means that interest charged in a period does not influence interest charged in the following one.
- In each period, interest will be computed multiplying the principal by the interest rate.
- Interest charged are only proportioned to the time of the investment.

….

Simple Rate

- If the time has a value, a loan's value changes over time.

Suppose for simplicity that:

- Yearly interest are computed at the end of each period;
- The principal, as well as all the charged interest, is paid at the end of the last period (that is the n-th period)
- The sum of all the interest and the principal represent the **future value** of the loan.

Simple Rate

- At the end of the first period the value of the loan will be:

$$P + P*r = \mathbf{P*(1+r)}$$

- At the end of the second period the value of the loan will be:

$$P + P*r + P*r = P+2P*r = \mathbf{P*(1+2r)}$$

- At the end of the generic t-th period the value of the loan will be:

$$\mathbf{P*(1+t*r)}$$

- At the end of the last period, the n-th, the value of the loan will be:

$$\mathbf{P*(1+n*r)}$$

Simple Rate

Suppose a loan with P = 100, r = 5% and n = 10.

- At the end of the first year, the value of the loan will be: 100*(1+5%) = $105; (…$100 today equals $105 next year!)
- In t=2, the value will be: 100 + 100*5% + 100*5% = 100*(1+2*5%) = $110;

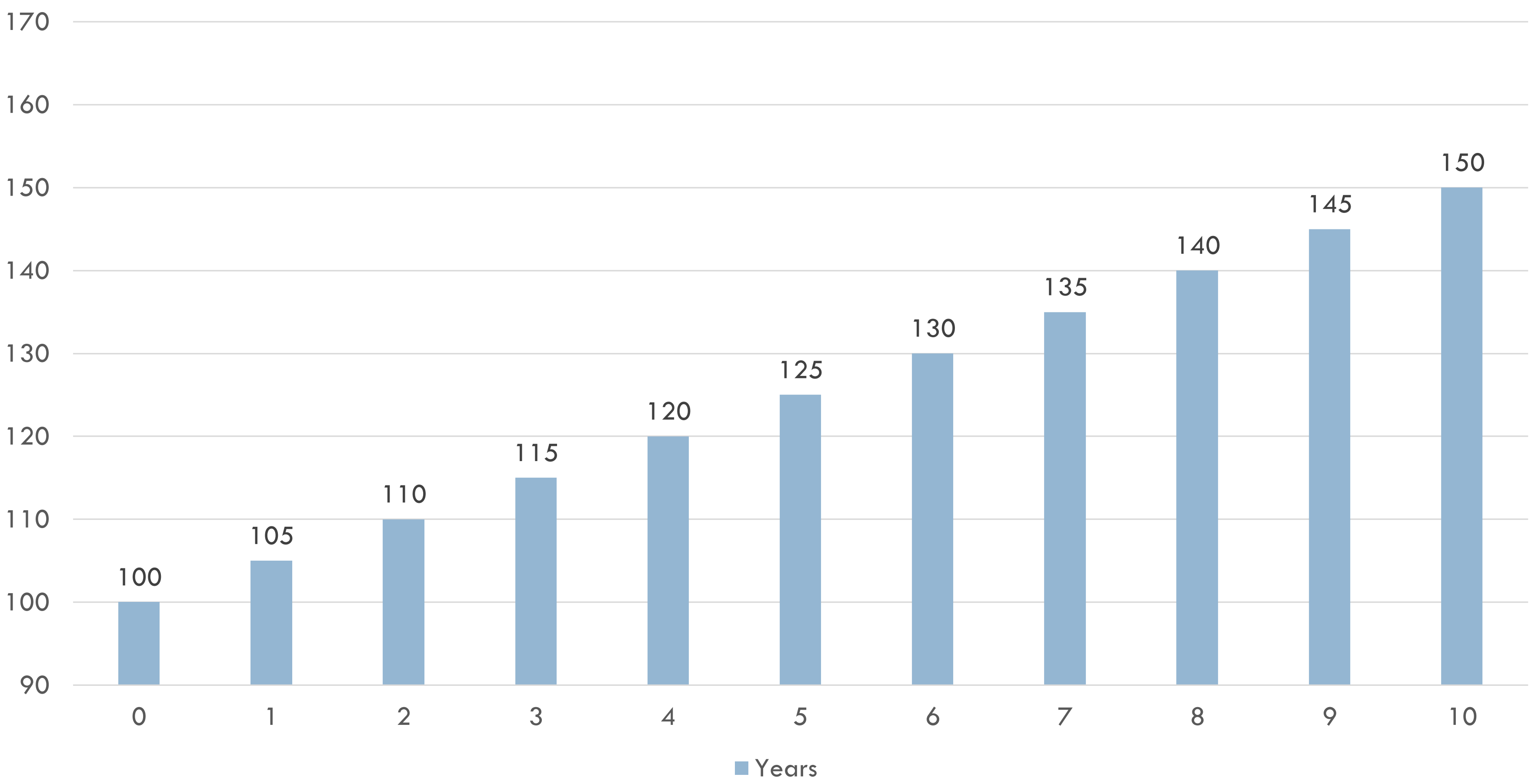
…

- In t=5, the value will be: 100*(1+5*5%) = $125

…

- In t=10, the value of the loan will be: 100*(1+10*5%) = $150

Simple Rate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10
Years

Simple Rate

- However, on the market one does not observe directly interest rates quotes;
- One can indeed observe Price and Future Value;
- By combining these two information, one can easily compute the interest rate;

Simple Rate

- Knowing the generic formula for the future value, according to the simple rate:

$$FV = P * (1 + t * r)$$

- One can make explicit the interest rate r, by inverting the formula hence obtaining:

$$r = (\frac{FV}{P} - 1)/t$$

Simple Rate

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Exercise

- Basing on the information contained in the table below, please fill the gaps

Values as of January 28, 2016	Values as of January 28, 2018	Annual rate
€ 7,000.00		5%
	€ 12,000.00	3%
€ 15,000.00	€ 17,500.00	

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Compound Rate

- According to the Compund Rate, Interest on period "*t*" are computed with respect to the value of the principal in period "*t-1*".
- It means that interest charged in a period influences interest charged in the following one.
- In each period, contrarily to the simple rate, interest will not be computed multiplying the principal by the interest rate.

….

Compound Rate

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…once again…

Knowing the time value of money, a loan's value changes over time.

Suppose for simplicity that:

- Yearly interest are computed at the end of each period;
- The principal, as well as all the charged interest, is paid at the end of the last period (that is the n-th period)
- The sum of all the interest and the principal represent the **future value** of the loan.

Compound Rate

- At the end of the first period the value of the loan will be:

$$P + P*r = \mathbf{P*(1+r)}$$

- At the end of the second period the value of the loan will be:

$$P*(1+r) + P*(1+r)*r = P*(1+r)*(1+r) = \mathbf{P*(1+r)^2}$$

- At the end of the generic t-th period the value of the loan will be:

$$P*(1+r)^{(t-1)} + [P*(1+r)^{(t-1)}]*r = P*(1+r)^{(t-1)}*(1+r) = \mathbf{P*(1+r)^t}$$

- At the end of the last period, the n-th, the value of the loan will be:

$$\mathbf{P*(1+r)^n}$$

Compound Rate

Suppose a loan with P = 100, r = 5% and n = 10.

- At the end of the first year, the value of the loan will be: 100*(1+5%) = $105; (…the value of $100 now equals $105 next year!)
- In t=2, the value will be: 100*(1+5%) + 100*(1+5%)*5% = $100(1+5\%)^2$ = $110.25;

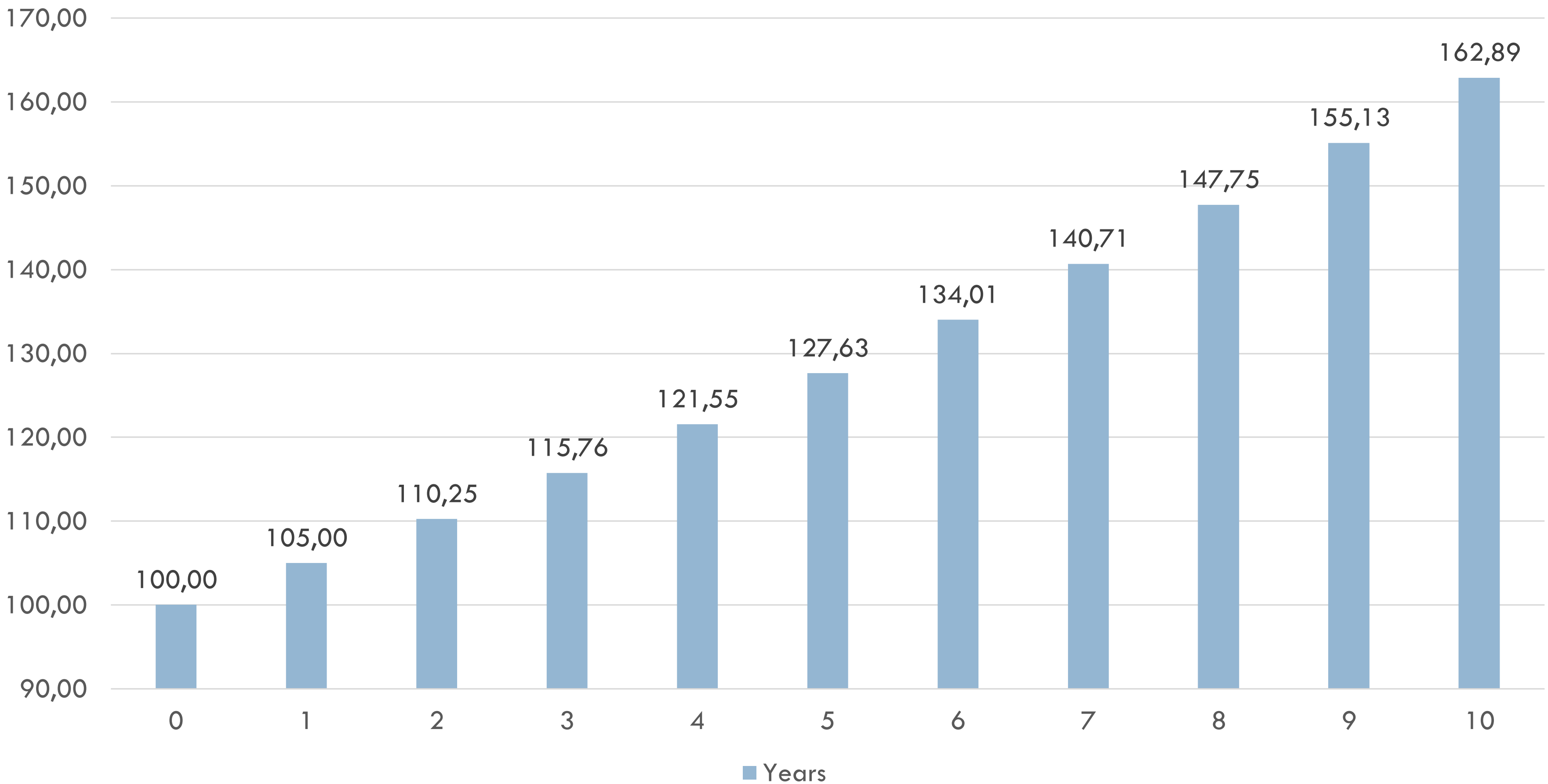
…

- In t=5, the value will be: 100*(1+5%) *(1+5%) *(1+5%) *(1+5%) *(1+5%) = $100*(1+5\%)^5$ = $127.63

…

- In t=10, the value of the loan will be: $100*(1+5\%)^{10}$ = $162.89

Compound Rate



Compound Rate

- As for the simple rate, compound rate can be computed starting by the
- Knowing the generic formula for the future value, according to the compound rate:

$$FV = P * (1 + r)^t$$

- One can make explicit the interest rate r, by inverting the formula hence obtaining:

$$r = \sqrt[t]{\frac{FV}{P}} - 1$$

Compound Rate

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Exercise

- Basing on the information contained in the table below, please fill the gaps

Values as of January 28, 2016	Values as of January 28, 2018	Annual rate
€ 14,000.00		6%
	€ 9,000.00	2.5%
€ 11,000.00	€ 13,500.00	

Compound Rate

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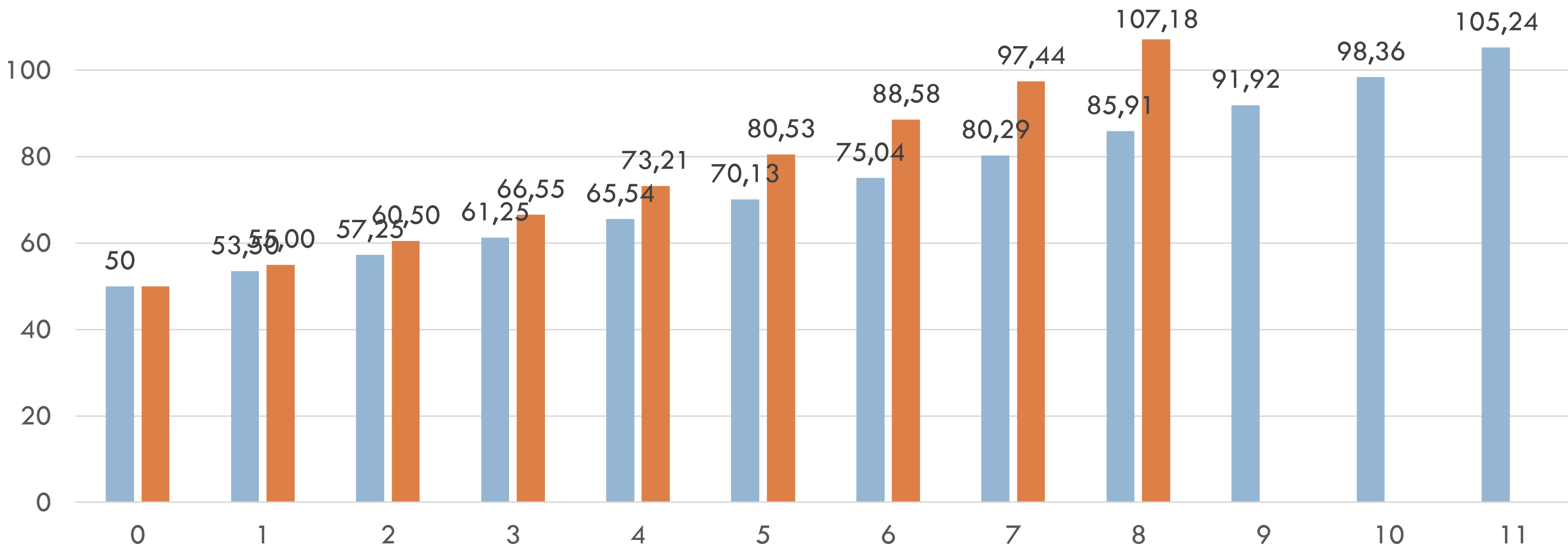
Focus: the seven-ten rule

- Money invested at 7% per year doubles in approximately 10 years. Also money invested at 10% per year doubles in approximately 7 years.



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Continuous Rate

- The idea behind the continuous rate is to charge interest on principal constantly over time.
- Imagine to divide a year into smaller and smaller periods.
- Interest charged in a period influences interest charged in the following one.
- In each period, contrarily to the simple rate, interest will not be computed multiplying the principal by the interest rate.

….

Continuous Rate

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Suppose that:

- You put your wealth into a bank account;
- Interest, 5%, are continuously compounded;
- You leave the principal, as well as all the charged interest, into the account at least for 10 years.

Continuous Rate

- Using the math vocabulary, the smallest part of a year can be written as:

$$\lim_{m \to \infty} \left(\frac{1}{m}\right)$$

- According to the previous formula, the number of sub-periods of period "*t*" goes to infinity.
- In each sub-period the value of the bank account is computed as:

$$\lim_{m \to \infty} \left(1 + \frac{r}{m}\right)^m = e^r$$, with e=2.7818….

Continuous Rate

- At the end of the first period the value of the bank account will be:

$$P*e^{1r}$$

- At the end of the 2^{nd} period the value of the bank account will be:

$$P*e^{2r}$$

- At the end of the generic i^{th} period the value of the account will be:

$$P*e^{ir}$$

- At the end of the last period, the bank account will values:

$$P*e^{nt}$$

Continuous Rate

Suppose a loan with P = 100, r = 5% and n = 10.

- At the end of the first year, the value of the bank account will be: $100*e^{5\%} = 105.13$
- In t=2, the value will be: $100*e^{5\%*2} = 110.52$

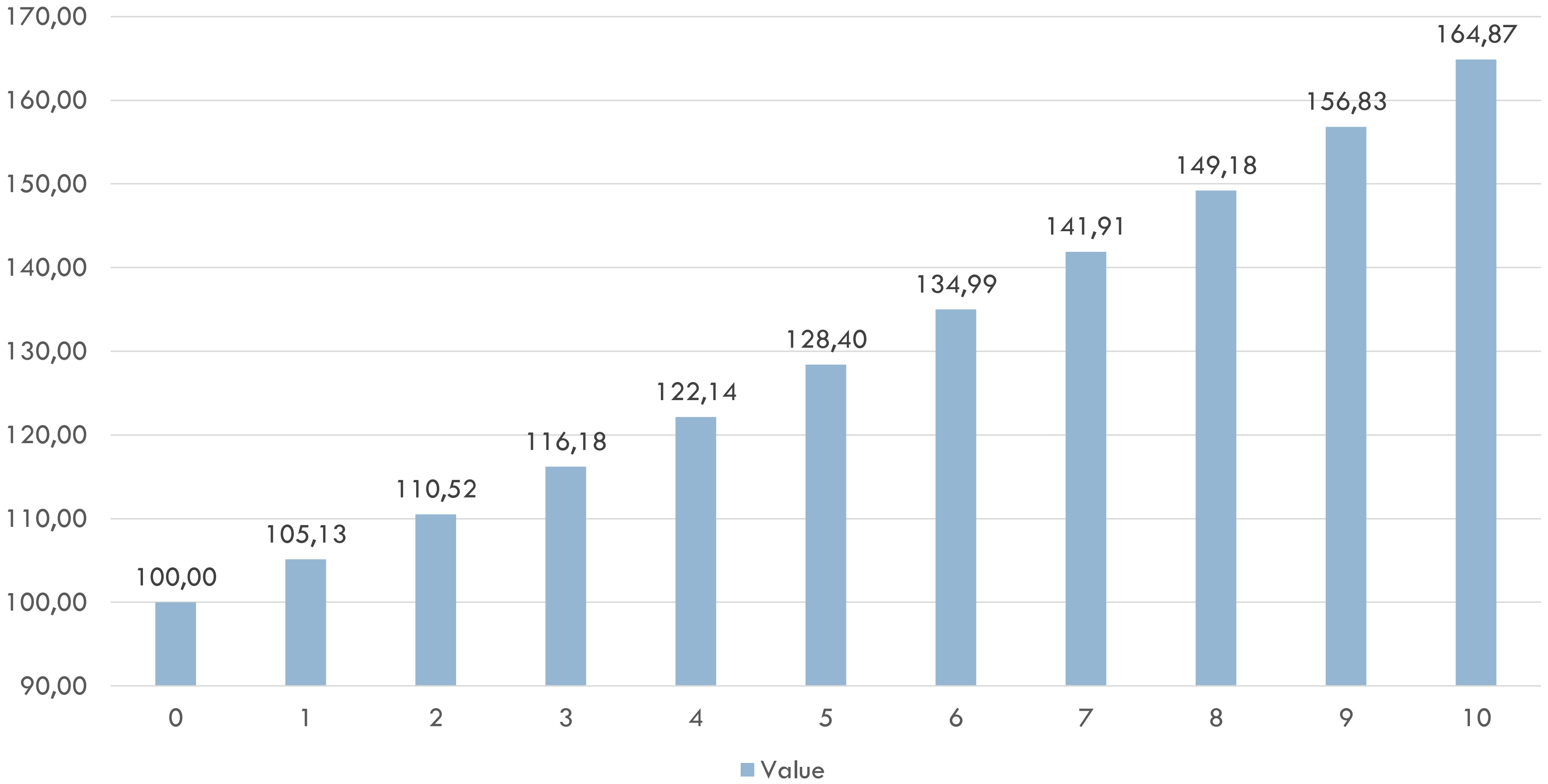
…

- In t=5, the value will be: $100*e^{5\%*5} = 128.40$

…

- In t=10, the value of the loan will be: $100*e^{5\%*10} = 164.87$

Continuous Rate



Continuous Rate

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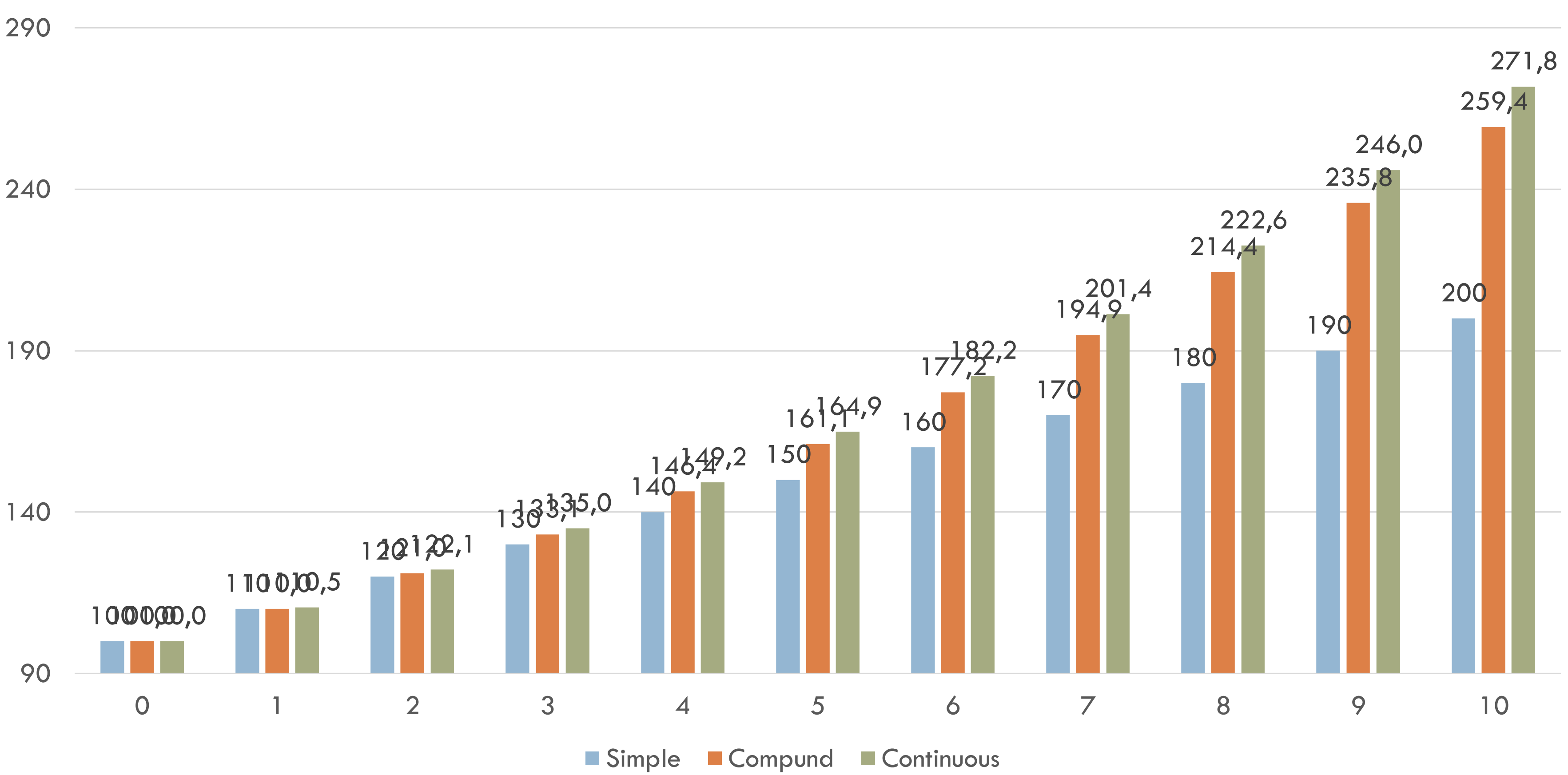
- Even continuous rate can be computed knowing present and future values.
- Starting by the generic formula for the future value:

$$FV = P * e^{r*t}$$

- One can make explicit the continuous interest rate r, by inverting the formula hence obtaining:

$$r = \frac{\ln\left(\frac{FV}{P}\right)}{t}$$

Interest rate regimes



Interest rate regimes

- According to the simple rate, the invested/borrowed money grows linearly with time.
- Under the compound interest, money exhibit a geometric growth.
- Continuous compounding leads to the familiar exponential growth curve.

Interest rates: focus

- Interest rate and time to maturity must be expressed on the same basis;
- That is, if the interest rate is expressed on a yearly basis time must be expressed in years.
- For example, in the future value formula,

$$FV = P * (1 + r)^t$$

Time (t) and interest rate (r) must be expressed on the same time frequency (year/year, month/month…)

Interest rates: focus

- What if the interest rate is computed yearly, while the time is a fraction (a quarter) or an imperfect multiple (e.g. 3 semester)?
- There can be two solutions:
  - Fraction and imperfect multiple can always be expressed in year. A quarter is 0.25 years; 3 semesters are 1.5 years, and so on and so forth…
  - Interest rates can be rescaled on the frequency of the maturity. If the maturity is a quarter, one can convert the annual rate into a quarterly rate.

Interest rates: focus

- Two rates are said to be equivalent if, for the same initial investment and **over the same time interval**, the final value of the investment, calculated with the two interest rates, is equal.
- Suppose the yearly r_Y and quarterly r_Q interest rate.
- There must be an equivalence between the future value of x dollars invested for a years. In formula:

$$FV_Q = x * \left(1 + r_Q\right)^4 = x * (1 + r_Y)^1 = FV_Y$$

Interest rates: focus

- Therefore, to convert an annual rate into quarterly rate the following equation must hold

¼ is the number of year in a quarter

$$(1+r_Q)^1 = (1+r_Y)^{\frac{1}{4}} \longrightarrow r_Q = (1+r_Y)^{\frac{1}{4}} - 1$$

- To convert quarterly rate into yearly one the following condition must hold

4 is the number of quarter in a year

$$(1+r_Q)^4 = (1+r_Y)^1 \longrightarrow r_Y = (1+r_Q)^4 - 1$$

Interest rates: focus

Exercises:

Fill the gaps in the table below with equivalent interest rates.

1 month	1 quarter	1 semester	1 year
1%			
	2%		
		4%	
			6%

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Future Value

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- The theme of the previous slides is that money invested today leads to increased value in the future as a result of interest.
- Up to now we have considered what is the future value of a single investment made in time 0; that is to study the impact of interest on a single cash flow.
- However one can, and often do, invest money in several time periods, and hence constitute a cash flow stream.
- From now on, we consider the future value of a cash flow stream.

Future Value

- The following cash flow stream summarize the activity of a bank account, whose interest are computed at the interest rate *r*:

$$(x_0, x_1, x_2, \dots, x_t, \dots, x_n | 0, 1, 2, \dots, t, \dots, n\ )$$

- In period 0 (that is, today) one deposits the quantity x_0; this sum generates interest for n periods.
- In period 1, x_1 is deposited; it generates interest for n-1 periods
- The x_n sum, deposited in the last period does not generate any interest.

Future Value

- The final balance in the account can be computed by summing the future value of each individual flow.
- Using the continuous compounding interest rate, the future value of x_0 (FV_0) is: $FV_0 = x_0 * e^{r*n}$
- The future value of x_1 (FV_1) is: $FV_1 = x_1 * e^{r*(n-1)}$
- The future value of x_t (FV_t) is: $FV_t = x_t * e^{r*(n-t)}$
- The future value of x_n (FV_n) is: $FV_n = x_n * e^{r*(n-n)} = x_n * e^0 = x_n$

Future Value

Concluding,

- Given a cash slow stream $(x_0, x_1, x_2, \dots, x_i, \dots, x_n | 0, 1, 2, \dots, i, \dots, n\ )$
- Given an interest rate r
- The future value of the stream is

$$FV = x_0 * e^{r*n} + x_1 * e^{r*(n-1)} + x_2 * e^{r*(n-2)} + \cdots +$$
$$+ x_t * e^{r*(n-t)} + \cdots + x_n$$

Future Value

Exercise

- The activity of a bank account is the following one:

$$(\$100, \$100, \$100 \,|0, 1, 2\,)$$

- Suppose that the interest rate r is 5%.
- Compute the final balance in the account by using the compound interest rate.

…hint: the generic formula of the FV under the compound rate is

$$FV = x_t * (1 + r)^t$$

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Present Value

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- The present value is the value that should be assigned now, in the present, to money that is to be received at a later time.
- The present value can be computed by reversing the formulas of the Future Values used up to now.
- While the process of evaluating the Future Value (FV) is referred to as **capitalizing**, the process of evaluating the Present Value (PV) is referred to as **discounting**.

Present Value

- As for the FV, the formula for the Present Value depends on the interest rate, *r*.
- Knowing that under the simple rate regimes the FV = PV*(1+i*r), the PV of a generic monetary amount available in the i-th period can be computed by reversing the formula for the FV as it follows:

$$PV = \frac{FV_i}{1 + t * r}$$

Present Value

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Exercise

- Compute the generic formula for the present value under the hypothesis of compound interest rate.
- Compute the generic formula for the present value under the hypothesis of continuous interest rate.

…hint: generic FV in time i, are respectively $FV_t = PV * (1 + r)^t$ *and* $FV_t = PV * e^{r*t}$

Present Value

The present value of a single cash flow available is time *i is:*

Simple rate:

$$PV = \frac{FV_t}{1+t*r}$$

Compound rate:

$$PV = \frac{FV_i}{(1+r)^t} = FV_t * (1+r)^{-t}$$

Continuous rate:

$$PV = \frac{FV_t}{e^{t*r}} = FV_t * e^{-(t*r)}$$

Present Value

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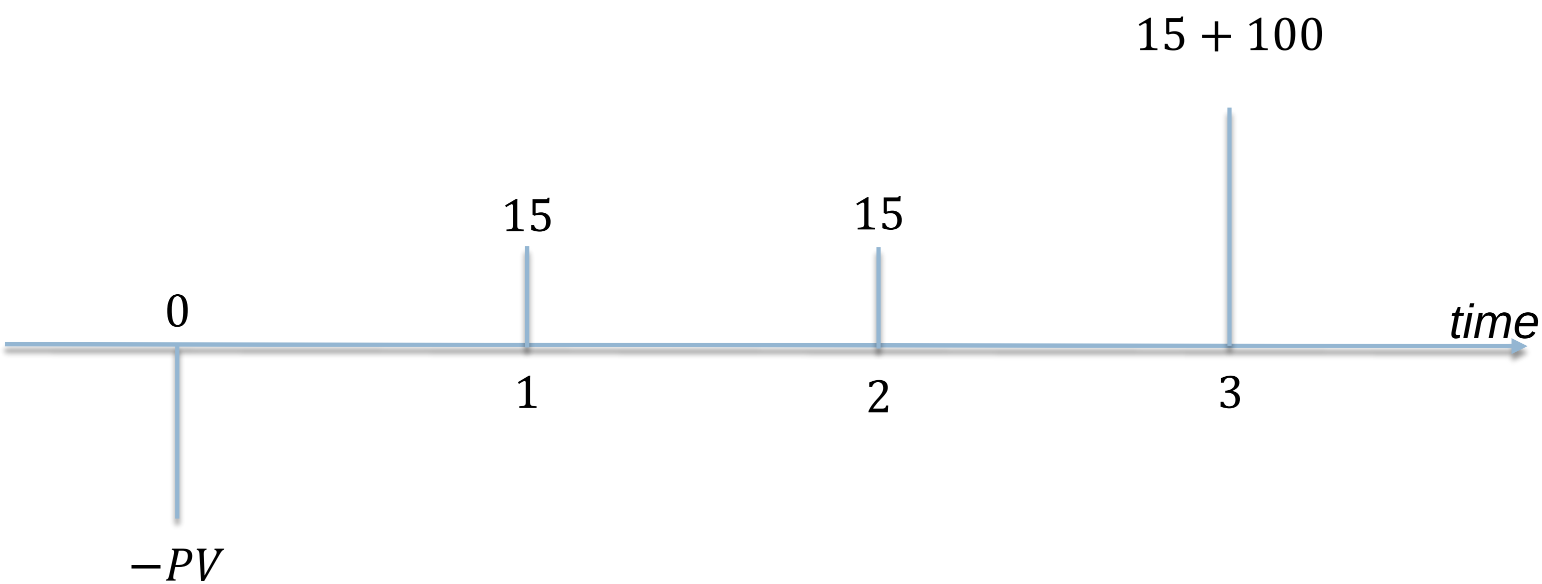
- Many situations impose to compute the present value of a cash flow stream;
- For example, suppose a coupon bond whose features are:
  - 15 euros of yearly coupon;
  - 3 years to maturity;
  - Interest rate = 5%;
  - Face value = € 100.
- Any potential investor must compute the present value of the bond, before buying it.

Present Value

Cash flow stream of coupon bond

- $(Price, x_2, x_3, \dots, -x_i, \dots, x_n | t_1, t_2, t_3, \dots, t_i, \dots, t_n)$



Present Value

- The present value of the bond equals:

The present value of the first coupon: $PV_1 = \frac{15_1}{(1+5\%)^1}$

+

The present value of the second coupon: $PV_2 = \frac{15_2}{(1+5\%)^2}$

+

The present value of the third coupon: $PV_3 = \frac{15_3}{(1+5\%)^3}$

+

The present value of the principal: $PV_3 = \frac{100_3}{(1+5\%)^3}$

Present Value

- Summarizing…

$$Price = \frac{15_1}{(1+5\%)^1} + \frac{15_2}{(1+5\%)^1} + \frac{115_3}{(1+5\%)^3}$$

$$Price = €\,14.29 + €\,13.61 + €\,99.34 = €\,127.23$$

Sum up and conclusion

- A dollar today is worth more than a dollar tomorrow.
- Time value of money is expressed concretely as an interest rate.
- Interest is the price paid for borrowing money.
- Interest rate it is interest expressed as percentage of the principal.
- Present Value is the discounted magnitude of a cash flow available at a future date.
- Future Value is the capitalized magnitude of a cash flow available at a present date.
- Cash flow are the amounts of money that will flow to and from an investor over time.
- Cash flow stream is a series of cash flows over several periods.