

INTEREST RATES

Luigi Vena

February 19, 2018

Liuc – Carlo Cattaneo

TODAY'S AGENDA

- ❑ Course structure
- ❑ Finance Dictionary
- ❑ Simple Rate
- ❑ Compound Rate
- ❑ Continuous Rate
- ❑ Future Value
- ❑ Present Value



International Financial Markets – 02/19/2018

Mishkin, Eakins – ch. 3-4

Instructors

Prof. Marcello Esposito

E-mail: mesposito@liuc.it
Office: “Torre” (main tower), 7th floor
Phone: 0331 572 205

Luigi Vena, PhD

E-mail: lvena@liuc.it
Office: “Torre” (main tower), 4th floor
Phone: 0331 572 250
Office hours: whenever you want (An **appointment via email must be previously set**).

Student's Assessment

Students attending the course (at least 9 classes):

- Home assignments (25%)
- Research paper (25%)
- Written exam (50%)

Students not attending the course:

- Written exam.

REQUIRED

- Frederic S. Mishkin, Stanley Eakins (2015). Financial Markets and Institutions, 8/E, Pearson. (available also in Italian: Frederic S. Mishkin, Stanley Eakins, Forestieri G. (2015). Istituzioni e Mercati Finanziari, 8/E, Pearson)

SUGGESTED

- Brealey, R. A., Myers, S. C., & Allen, F. (2014). Principles of corporate finance. New York, NY, McGraw-Hill/Irwin.
- Charles P. Kindleberger, A Financial History of Western Europe (London: Routledge 2007). Chapters: 1 (19-34); 4 (55-70); 15-16

TODAY'S AGENDA

- ❑ Course structure
- ❑ **Finance Dictionary**
- ❑ Simple Rate
- ❑ Compound Rate
- ❑ Continuous Rate
- ❑ Future Value
- ❑ Present Value



International Financial Markets – 02/19/2018

Mishkin, Eakins – ch. 3-4

Interest and interest rates

- Interest: amount of money charged by a lender to a borrower for the use of assets;
- Interest rate: is the interest expressed as percentage of the principal.

Example:

- Principal = 100\$ and Interest = 10\$ \rightarrow Interest rate = $10\$/100\$ = 10\%$
- Principal = 100\$ and Interest rate = 15% \rightarrow Interest = $100\$ * 15\% = 15\$$

Interest rates

- The 1-year interest rate represents the price paid (as percentage of the principal) for borrowing money in a year.
- Interest rate can be computed at any frequency, not just yearly.
- Interest rate is simply the cost of borrowing or the price paid for the rental of fund.

I principle of finance: *a dollar today is worth more than a dollar tomorrow.*

- Money can be invested to earn interest.
- Between \$100 now and \$100 next year, one takes the money now to get a year's interest.

Future Value vs Present Value

- **Future Value:** The value of cash at a specified date in the future that is equivalent in value to a specified sum today.
- **Present Value:** the value that should be assigned now, in the present, to money that is to be received at a later time.

Future Value vs Present Value

- Money received in the future is worth less than the same amount of money received in the present.
- Money received today can be invested to earn interest.
- **Present value** is the discounted magnitude of a cash flow available at a future date.
- **Future value** is the capitalized magnitude of a cash flow available in the present.

From now on, we use the following notation:

P, to indicate the principal i.e.:

- The face value of a bond;
- The amount borrowed or the amount still owed on a loan;
- The original amount invested.

r, to indicate the interest rate;

I, to indicate the interest;

n, total number of periods.

t, the time (usually expressed in years)

Cash Flow and Cash Flow Stream

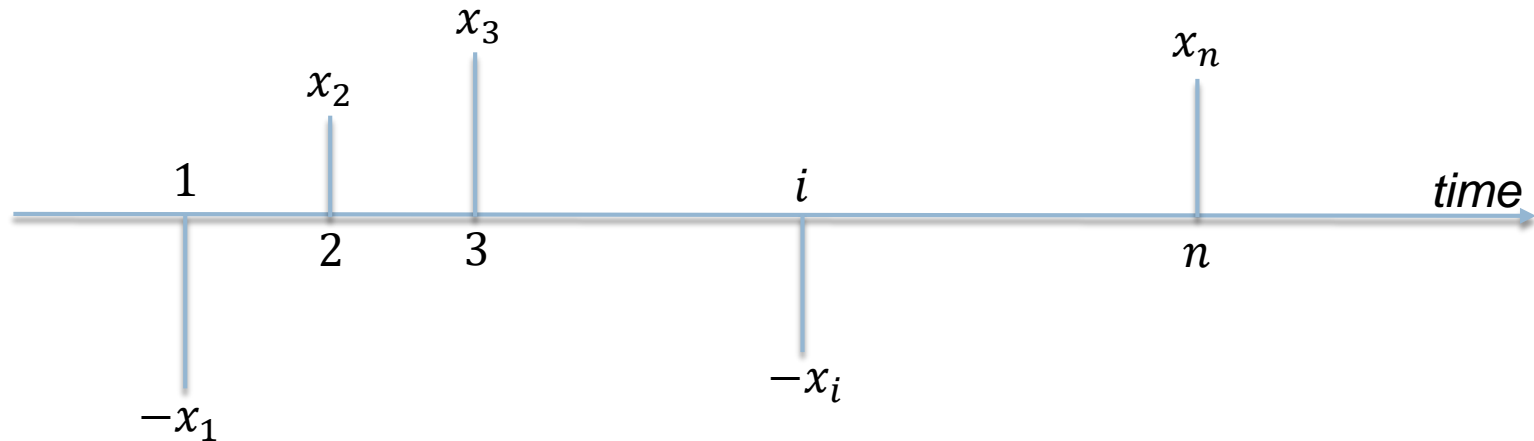
- Cash flows are the amounts of money that will flow to and from an investor over time.
- Cash flows (either positive or negative) occur at a known specific dates, such as at the end of each month/quarter/year.
- The stream of cash flow can be described by listing flows at each of the date in which they occur.

Cash Flow and Cash Flow Stream

- Among others, cash flow stream can be represented by a diagram, where:
- Negative cash flows represent cash outlays.
- Positive cash flows represent cash collections/proceeds.

Cash Flow and Cash Flow Stream Representation

□ $(-x_1, x_2, x_3, \dots, -x_i, \dots, x_n | t_1, t_2, t_3, \dots, t_i, \dots, t_n)$



TODAY'S AGENDA

- ❑ Course structure
- ❑ Finance Dictionary
- ❑ **Simple Rate**
- ❑ Compound Rate
- ❑ Continuous Rate
- ❑ Future Value
- ❑ Present Value



International Financial Markets – 02/19/2018

Mishkin, Eakins – ch. 3-4

Simple Rate

- According to the Simple Rate, Interest are only computed with respect to the principal.
- It means that interest charged in a period does not influence interest charged in the following one.
- In each period, interest will be computed multiplying the principal by the interest rate.
- Interest charged are only proportioned to the time of the investment.

....

Simple Rate

- If the time has a value, a loan's value changes over time.

Suppose for simplicity that:

- Yearly interest are computed at the end of each period;
- The principal, as well as all the charged interest, is paid at the end of the last period (that is the n -th period)
- The sum of all the interest and the principal represent the **future value** of the loan.

Simple Rate

- At the end of the first period the value of the loan will be:

$$P + P*r = \mathbf{P*(1+r)}$$

- At the end of the second period the value of the loan will be:

$$P + P*r + P*r = P+2P*r = \mathbf{P*(1+2r)}$$

- At the end of the generic t-th period the value of the loan will be:

$$\mathbf{P*(1+t*r)}$$

- At the end of the last period, the n-th, the value of the loan will be:

$$\mathbf{P*(1+n*r)}$$

Simple Rate

Suppose a loan with $P = 100$, $r = 5\%$ and $n = 10$.

□ At the end of the first year, the value of the loan will be: $100 \cdot (1 + 5\%) = \$105$;
(...\$100 today equals \$105 next year!)

□ In $t=2$, the value will be: $100 + 100 \cdot 5\% + 100 \cdot 5\% = 100 \cdot (1 + 2 \cdot 5\%) = \110 ;

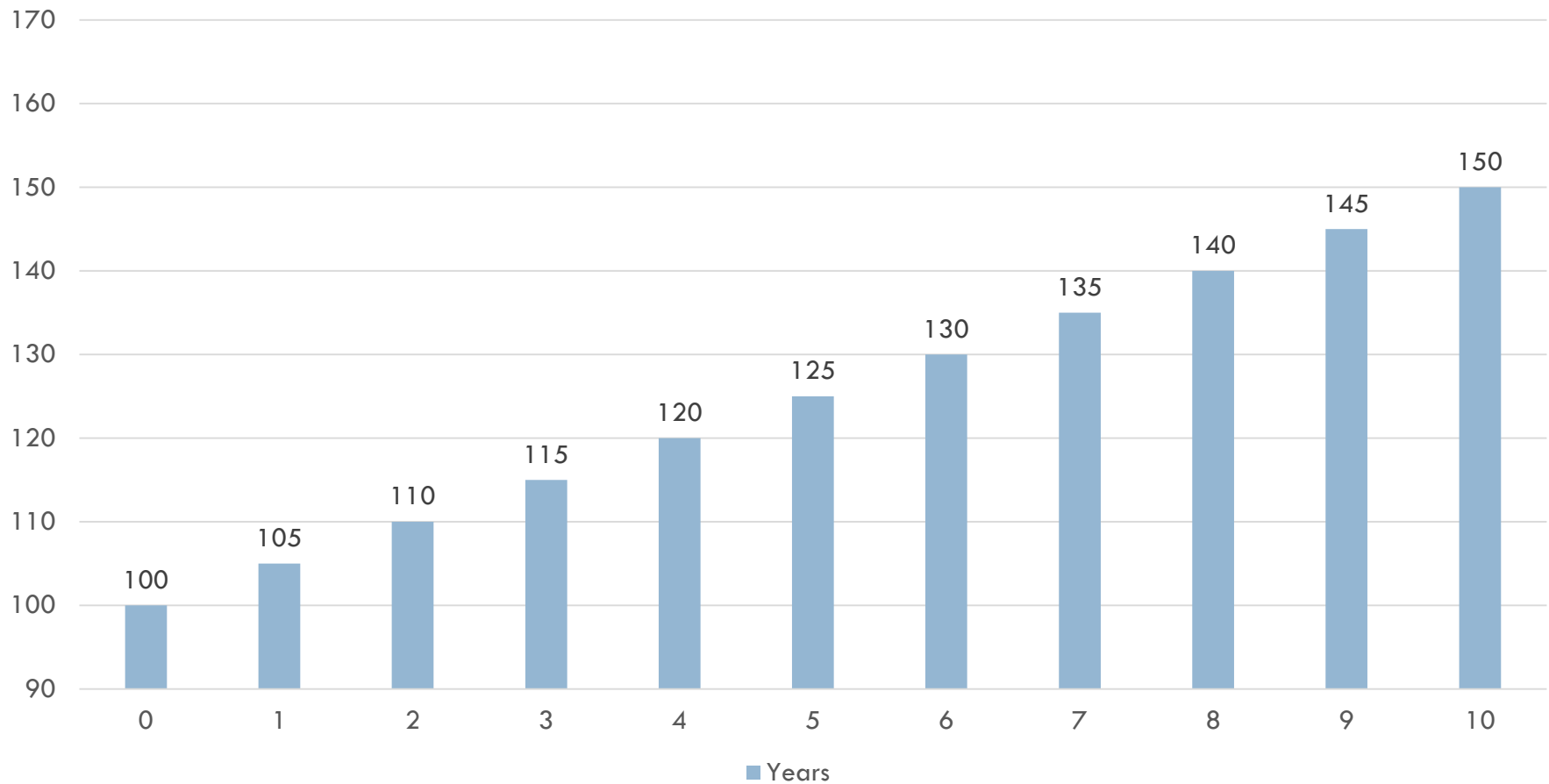
...

□ In $t=5$, the value will be: $100 \cdot (1 + 5 \cdot 5\%) = \125

...

□ In $t=10$, the value of the loan will be: $100 \cdot (1 + 10 \cdot 5\%) = \150

Simple Rate



Simple Rate

- However, on the market one does not observe directly interest rates quotes;
- One can indeed observe Price and Future Value;
- By combining these two information, one can easily compute the interest rate;

Simple Rate

- Knowing the generic formula for the future value, according to the simple rate:

$$FV = P * (1 + t * r)$$

- One can make explicit the interest rate r , by inverting the formula hence obtaining:

$$r = (\frac{FV}{P} - 1)/t$$

Simple Rate

Exercise

- Basing on the information contained in the table below, please fill the gaps

Values as of January 28, 2016	Values as of January 28, 2018	Annual rate
€ 7,000.00		5%
	€ 12,000.00	3%
€ 15,000.00	€ 17,500.00	

TODAY'S AGENDA

- ☐ Course structure
- ☐ Finance Dictionary
- ☐ Simple Rate
- ☐ **Compound Rate**
- ☐ Continuous Rate
- ☐ Future Value
- ☐ Present Value



International Financial Markets – 02/19/2018

Mishkin, Eakins – ch. 3-4

Compound Rate

- According to the Compound Rate, Interest on period “ t ” are computed with respect to the value of the principal in period “ $t-1$ ”.
- It means that interest charged in a period influences interest charged in the following one.
- In each period, contrarily to the simple rate, interest will not be computed multiplying the principal by the interest rate.

....

Compound Rate

...once again...

Knowing the time value of money, a loan's value changes over time.

Suppose for simplicity that:

- Yearly interest are computed at the end of each period;
- The principal, as well as all the charged interest, is paid at the end of the last period (that is the n -th period)
- The sum of all the interest and the principal represent the **future value** of the loan.

Compound Rate

- At the end of the first period the value of the loan will be:

$$P + P*r = \mathbf{P*(1+r)}$$

- At the end of the second period the value of the loan will be:

$$P*(1+r) + P*(1+r)*r = P*(1+r)*(1+r) = \mathbf{P*(1+r)^2}$$

- At the end of the generic t-th period the value of the loan will be:

$$P*(1+r)^{(t-1)} + [P*(1+r)^{(t-1)}]*r = P*(1+r)^{(t-1)}*(1+r) = \mathbf{P*(1+r)^t}$$

- At the end of the last period, the n-th, the value of the loan will be:

$$\mathbf{P*(1+r)^n}$$

Compound Rate

Suppose a loan with $P = 100$, $r = 5\%$ and $n = 10$.

□ At the end of the first year, the value of the loan will be: $100 \cdot (1 + 5\%) = \$105$;
(...the value of \$100 now equals \$105 next year!)

□ In $t=2$, the value will be: $100 \cdot (1 + 5\%) + 100 \cdot (1 + 5\%) \cdot 5\% = 100(1 + 5\%)^2 = \110.25 ;

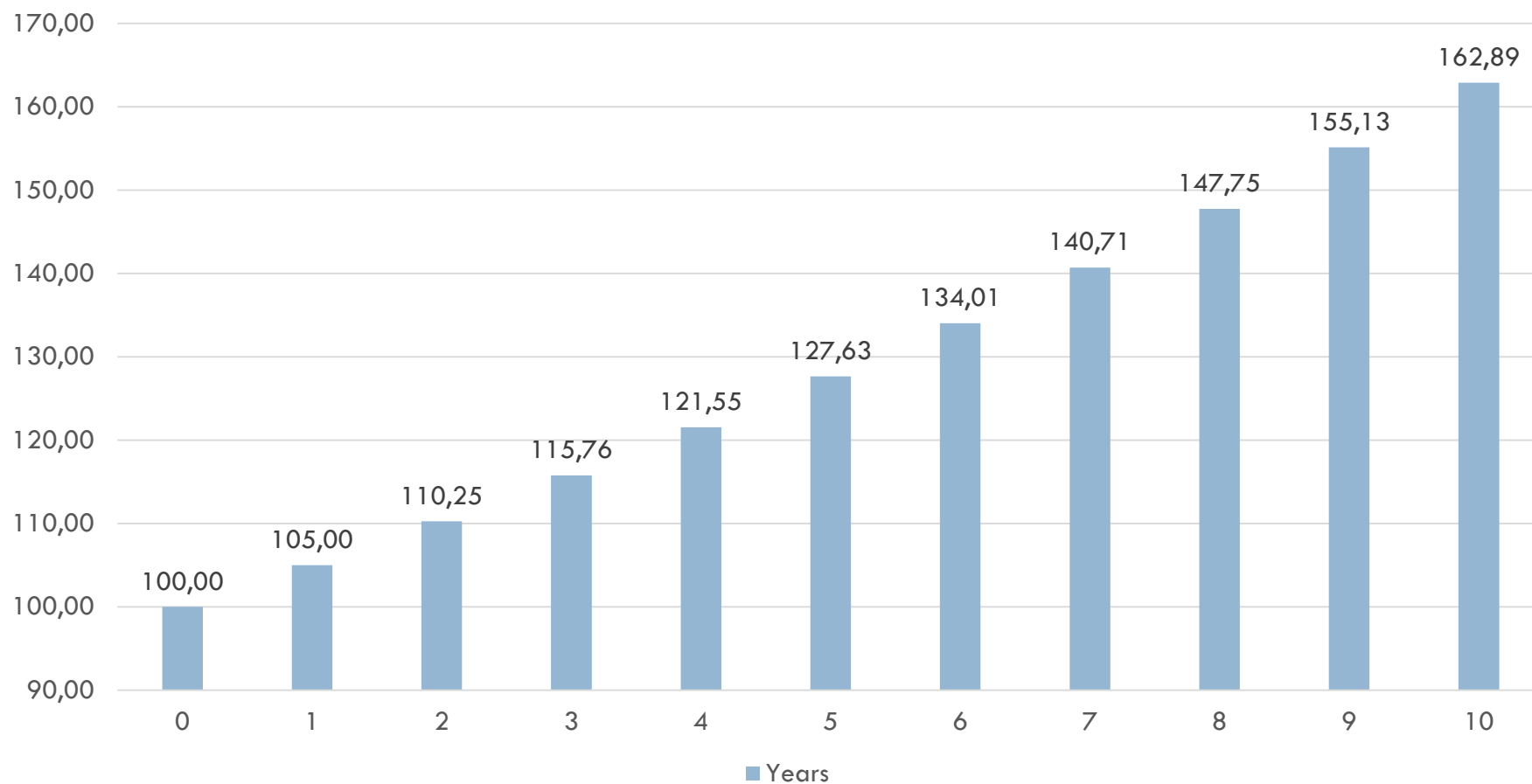
...

□ In $t=5$, the value will be: $100 \cdot (1 + 5\%) \cdot (1 + 5\%) \cdot (1 + 5\%) \cdot (1 + 5\%) \cdot (1 + 5\%) = 100 \cdot (1 + 5\%)^5 = \127.63

...

□ In $t=10$, the value of the loan will be: $100 \cdot (1 + 5\%)^{10} = \162.89

Compound Rate



Compound Rate

- As for the simple rate, compound rate can be computed starting by the
- Knowing the generic formula for the future value, according to the compound rate:

$$FV = P * (1 + r)^t$$

- One can make explicit the interest rate r , by inverting the formula hence obtaining:

$$r = \sqrt[t]{\frac{FV}{P}} - 1$$

Compound Rate

Exercise

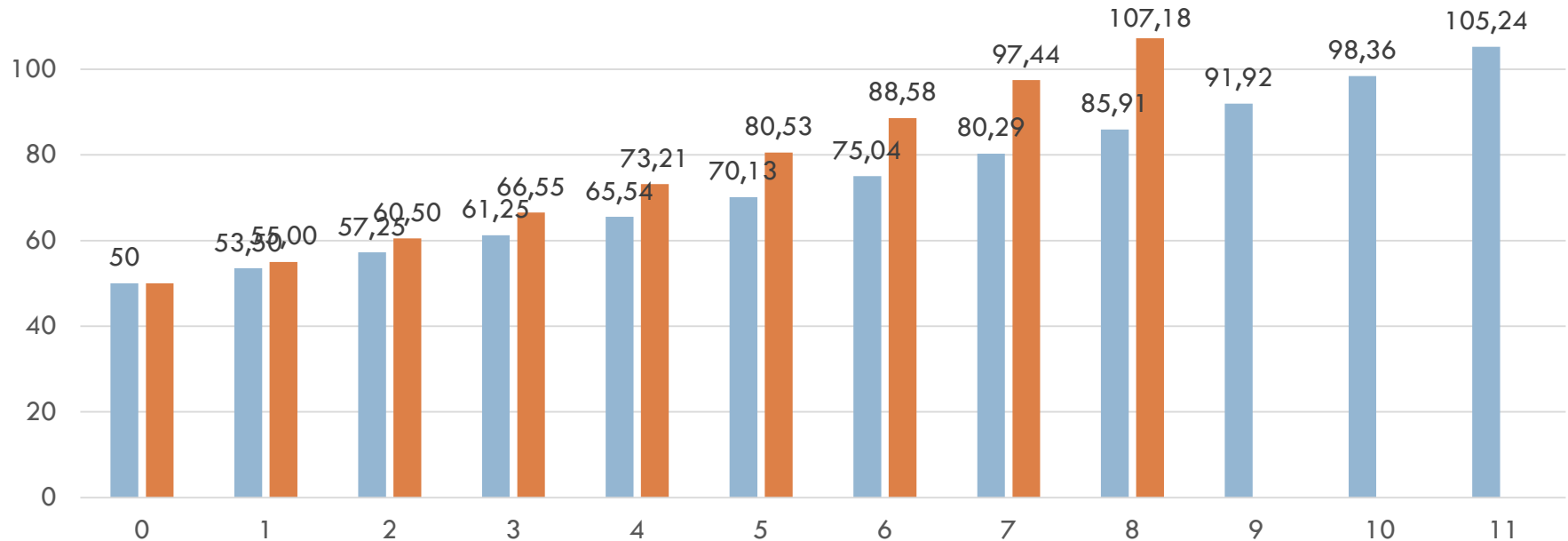
- Basing on the information contained in the table below, please fill the gaps

Values as of January 28, 2016	Values as of January 28, 2018	Annual rate
€ 14,000.00		6%
	€ 9,000.00	2.5%
€ 11,000.00	€ 13,500.00	

Compound Rate

Focus: the seven-ten rule

- Money invested at 7% per year doubles in approximately 10 years. Also money invested at 10% per year doubles in approximately 7 years.



TODAY'S AGENDA

- ☐ Course structure
- ☐ Finance Dictionary
- ☐ Simple Rate
- ☐ Compound Rate
- ☐ **Continuous Rate**
- ☐ Future Value
- ☐ Present Value



International Financial Markets – 02/19/2018

Mishkin, Eakins – ch. 3-4

Continuous Rate

- The idea behind the continuous rate is to charge interest on principal constantly over time.
- Imagine to divide a year into smaller and smaller periods.
- Interest charged in a period influences interest charged in the following one.
- In each period, contrarily to the simple rate, interest will not be computed multiplying the principal by the interest rate.

....

Continuous Rate

Suppose that:

- You put your wealth into a bank account;
- Interest, 5%, are continuously compounded;
- You leave the principal, as well as all the charged interest, into the account at least for 10 years.

- Using the math vocabulary, the smallest part of a year can be written as:

$$\lim_{m \rightarrow \infty} \left(\frac{1}{m} \right)$$

- According to the previous formula, the number of sub-periods of period “t” goes to infinity.
- In each sub-period the value of the bank account is computed as:

$$\lim_{m \rightarrow \infty} \left(1 + \frac{r}{m} \right)^{mt} = e^{rt}, \text{ with } e=2.7818....$$

Continuous Rate

- At the end of the first period the value of the bank account will be:

$$P * e^{1r}$$

- At the end of the 2nd period the value of the bank account will be:

$$P * e^{2r}$$

- At the end of the generic i^{th} period the value of the account will be:

$$P * e^{ir}$$

- At the end of the last period, the bank account will values:

$$P * e^{tr}$$

Continuous Rate

Suppose a loan with $P = 100$, $r = 5\%$ and $n = 10$.

□ At the end of the first year, the value of the bank account will be: $100 * e^{5\%} = 105.13$

□ In $t=2$, the value will be: $100 * e^{5\%*2} = 110.52$

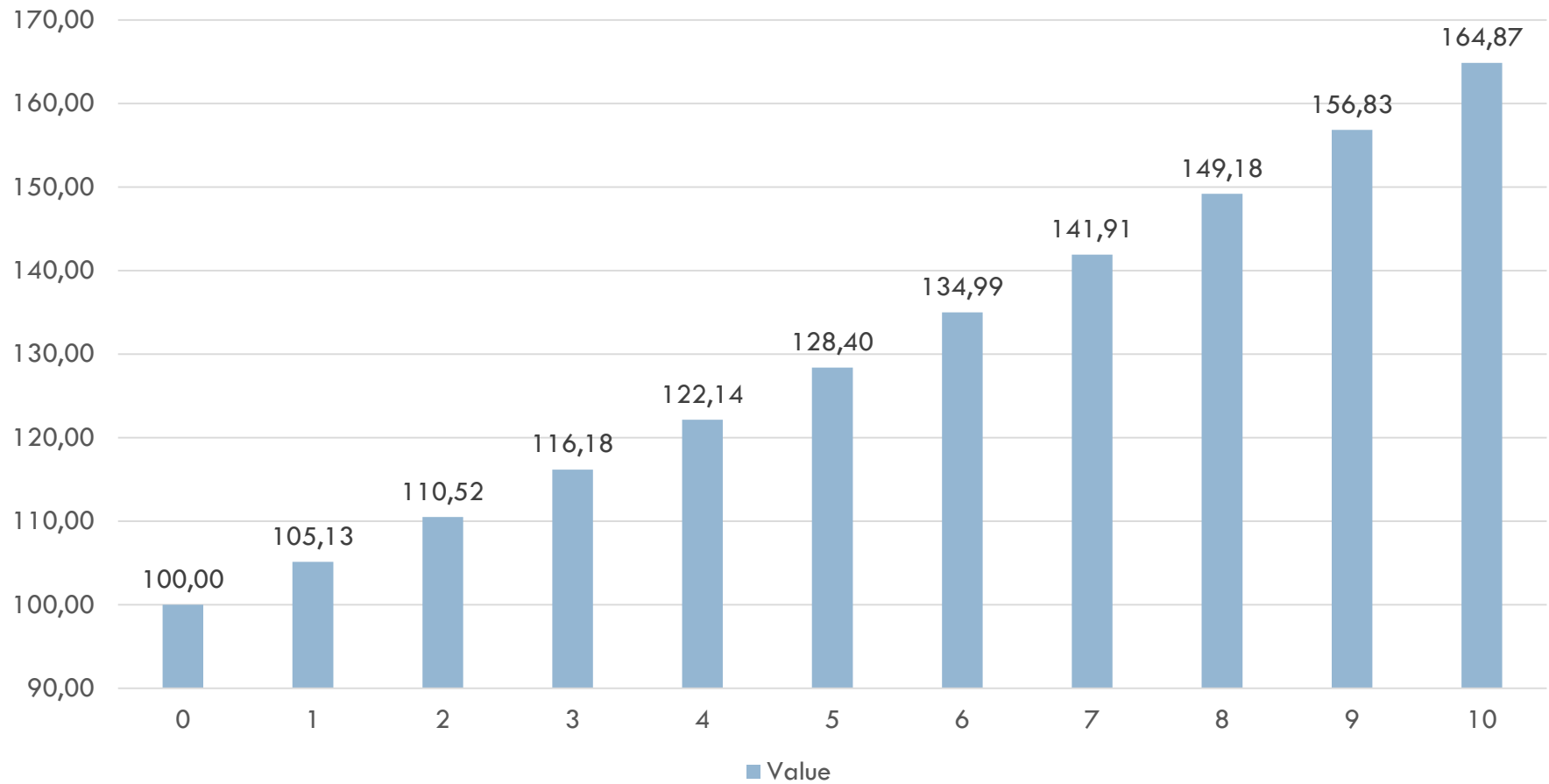
...

□ In $t=5$, the value will be: $100 * e^{5\%*5} = 128.40$

...

□ In $t=10$, the value of the loan will be: $100 * e^{5\%*10} = 164.87$

Continuous Rate



Continuous Rate

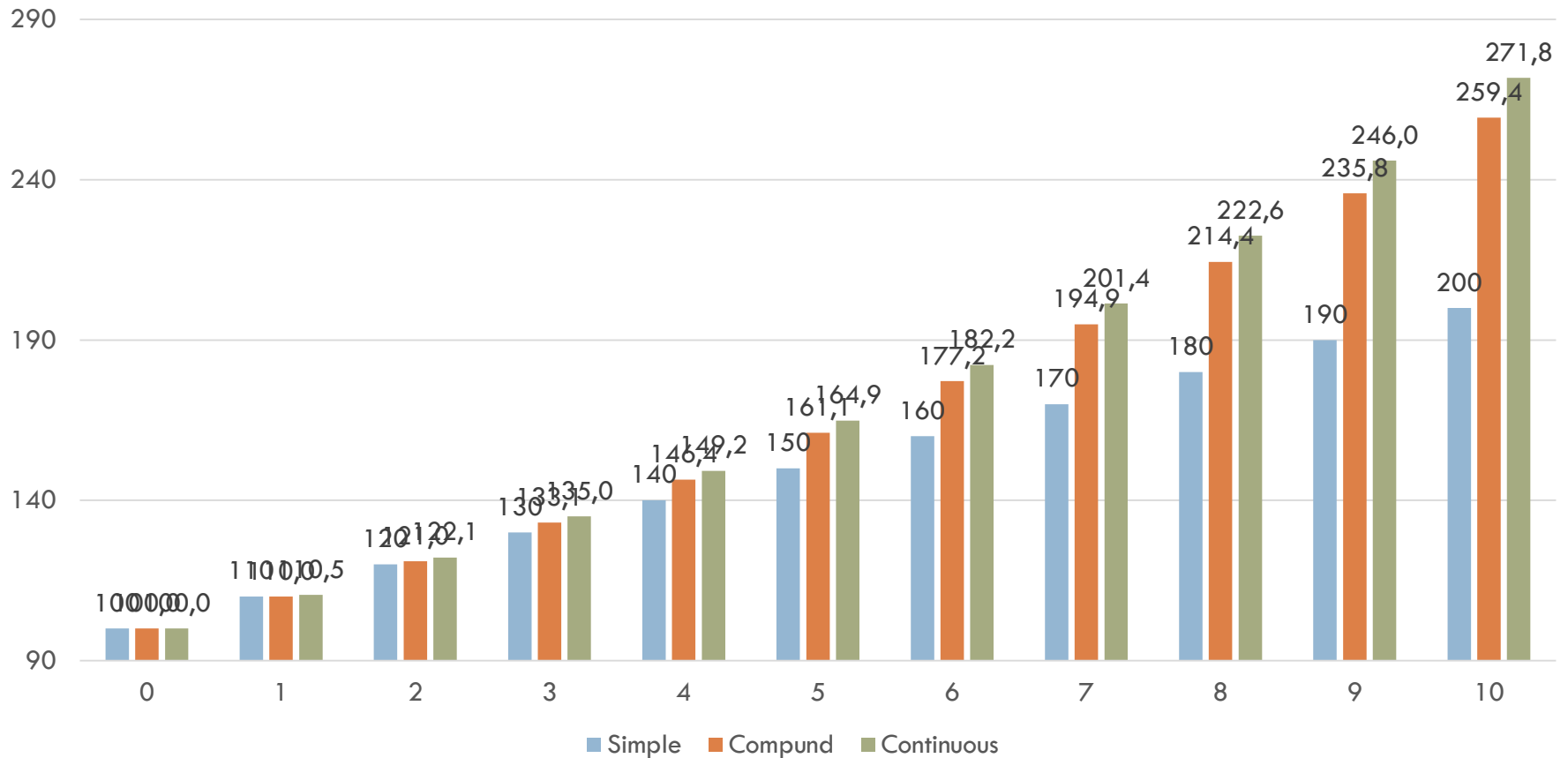
- Even continuous rate can be computed knowing present and future values.
- Starting by the generic formula for the future value:

$$FV = P * e^{r*t}$$

- One can make explicit the continuous interest rate r , by inverting the formula hence obtaining:

$$r = \frac{\ln\left(\frac{FV}{P}\right)}{t}$$

Interest rate regimes



Interest rate regimes

- According to the simple rate, the invested/borrowed money grows linearly with time.
- Under the compound interest, money exhibit a geometric growth.
- Continuous compounding leads to the familiar exponential growth curve.

Interest rates: focus

- Interest rate and time to maturity must be expressed on the same basis;
- That is, if the interest rate is expressed on a yearly basis time must be expressed in years.
- For example, in the future value formula,

$$FV = P * (1 + r)^t$$

Time (t) and interest rate (r) must be expressed on the same time frequency (year/year, month/month...)

Interest rates: focus

- What if the interest rate is computed yearly, while the time is a fraction (a quarter) or an imperfect multiple (e.g. 3 semester)?
- There can be two solutions:
 - ▣ Fraction and imperfect multiple can always be expressed in year. A quarter is 0.25 years; 3 semesters are 1.5 years, and so on and so forth...
 - ▣ Interest rates can be rescaled on the frequency of the maturity. If the maturity is a quarter, one can convert the annual rate into a quarterly rate.

Interest rates: focus

- Two rates are said to be equivalent if, for the same initial investment and **over the same time interval**, the final value of the investment, calculated with the two interest rates, is equal.
- Suppose the yearly r_Y and quarterly r_Q interest rate.
- There must be an equivalence between the future value of x dollars invested for a years. In formula:

$$FV_Q = x * (1 + r_Q)^4 = x * (1 + r_Y)^1 = FV_Y$$

Interest rates: focus

- Therefore, to convert an annual rate into quarterly rate the following equation must hold

$$(1 + r_Q)^1 = (1 + r_Y)^{\frac{1}{4}} \longrightarrow r_Q = (1 + r_Y)^{\frac{1}{4}} - 1$$

$\frac{1}{4}$ is the number of years in a quarter

- To convert quarterly rate into yearly one the following condition must hold

$$(1 + r_Q)^4 = (1 + r_Y)^1 \longrightarrow r_Y = (1 + r_Q)^4 - 1$$

4 is the number of quarters in a year

Interest rates: focus

Exercises:

Fill the gaps in the table below with equivalent interest rates.

1 month	1 quarter	1 semester	1 year
1%			
	2%		
		4%	
			6%

TODAY'S AGENDA

- ☐ Course structure
- ☐ Finance Dictionary
- ☐ Simple Rate
- ☐ Compound Rate
- ☐ Continuous Rate
- ☐ **Future Value**
- ☐ Present Value



International Financial Markets – 02/19/2018

Mishkin, Eakins – ch. 3-4

Future Value

- The theme of the previous slides is that money invested today leads to increased value in the future as a result of interest.
- Up to now we have considered what is the future value of a single investment made in time 0; that is to study the impact of interest on a single cash flow.
- However one can, and often do, invest money in several time periods, and hence constitute a cash flow stream.
- From now on, we consider the future value of a cash flow stream.

Future Value

- The following cash flow stream summarize the activity of a bank account, whose interest are computed at the interest rate r :

$$(x_0, x_1, x_2, \dots, x_t, \dots, x_n | 0, 1, 2, \dots, t, \dots, n)$$

- In period 0 (that is, today) one deposits the quantity x_0 ; this sum generates interest for n periods.
- In period 1, x_1 is deposited; it generates interest for $n-1$ periods
- The x_n sum, deposited in the last period does not generate any interest.

Future Value

- The final balance in the account can be computed by summing the future value of each individual flow.
- Using the continuous compounding interest rate, the future value of x_0 (FV_0) is: $FV_0 = x_0 * e^{r*n}$
- The future value of x_1 (FV_1) is: $FV_1 = x_1 * e^{r*(n-1)}$
- The future value of x_t (FV_t) is: $FV_t = x_t * e^{r*(n-t)}$
- The future value of x_n (FV_n) is: $FV_n = x_n * e^{r*(n-n)} = x_n * e^0 = x_n$

Concluding,

- Given a cash flow stream $(x_0, x_1, x_2, \dots, x_i, \dots, x_n | 0, 1, 2, \dots, i, \dots, n)$
- Given an interest rate r
- The future value of the stream is

$$FV = x_0 * e^{r*n} + x_1 * e^{r*(n-1)} + x_2 * e^{r*(n-2)} + \dots + \\ + x_t * e^{r*(n-t)} + \dots + x_n$$

Exercise

- The activity of a bank account is the following one:

$(\$100, \$100, \$100 \mid 0, 1, 2)$

- Suppose that the interest rate r is 5%.
- Compute the final balance in the account by using the compound interest rate.

...hint: the generic formula of the FV under the compound rate is

$$FV = x_t * (1 + r)^t$$

TODAY'S AGENDA

- ❑ Course structure
- ❑ Finance Dictionary
- ❑ Simple Rate
- ❑ Compound Rate
- ❑ Continuous Rate
- ❑ Future Value
- ❑ **Present Value**



International Financial Markets – 02/19/2018

Mishkin, Eakins – ch. 3-4

Present Value

- The present value is the value that should be assigned now, in the present, to money that is to be received at a later time.
- The present value can be computed by reversing the formulas of the Future Values used up to now.
- While the process of evaluating the Future Value (FV) is referred to as **capitalizing**, the process of evaluating the Present Value (PV) is referred to as **discounting**.

Present Value

- As for the FV, the formula for the Present Value depends on the interest rate, r .
- Knowing that under the simple rate regimes the $FV = PV * (1 + i * r)$, the PV of a generic monetary amount available in the i -th period can be computed by reversing the formula for the FV as it follows:

$$PV = \frac{FV_i}{1 + t * r}$$

Exercise

- Compute the generic formula for the present value under the hypothesis of compound interest rate.
- Compute the generic formula for the present value under the hypothesis of continuous interest rate.

*...hint: generic FV in time t , are respectively $FV_t = PV * (1 + r)^t$ and $FV_t = PV * e^{r*t}$*

Present Value

The present value of a single cash flow available at time i is:

Simple rate:

$$PV = \frac{FV_t}{1+t*r}$$

Compound rate:

$$PV = \frac{FV_t}{(1+r)^t} = FV_t * (1+r)^{-t}$$

Continuous rate:

$$PV = \frac{FV_t}{e^{t*r}} = FV_t * e^{-(t*r)}$$

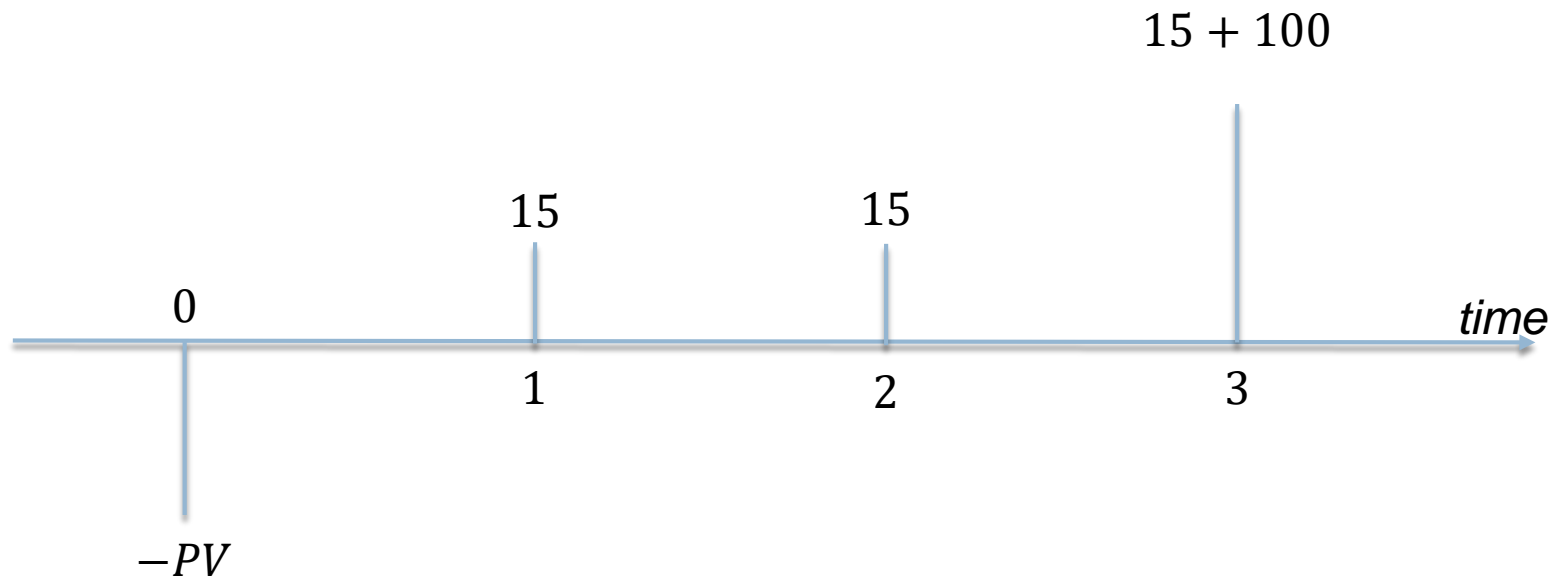
Present Value

- Many situations impose to compute the present value of a cash flow stream;
- For example, suppose a coupon bond whose features are:
 - ▣ 15 euros of yearly coupon;
 - ▣ 3 years to maturity;
 - ▣ Interest rate = 5%;
 - ▣ Face value = € 100.
- Any potential investor must compute the present value of the bond, before buying it.

Present Value

Cash flow stream of coupon bond

□ $(Price, x_2, x_3, \dots, -x_i, \dots, x_n | t_1, t_2, t_3, \dots, t_i, \dots, t_n)$



Present Value

- The present value of the bond equals:

$$\text{The present value of the first coupon: } PV_1 = \frac{15_1}{(1+5\%)^1}$$

+

$$\text{The present value of the second coupon: } PV_2 = \frac{15_2}{(1+5\%)^2}$$

+

$$\text{The present value of the third coupon: } PV_3 = \frac{15_3}{(1+5\%)^3}$$

+

$$\text{The present value of the principal: } PV_3 = \frac{100_3}{(1+5\%)^3}$$

□ Summarizing...

$$Price = \frac{15_1}{(1 + 5\%)^1} + \frac{15_2}{(1 + 5\%)^1} + \frac{115_3}{(1 + 5\%)^3}$$

$$Price = € 14.29 + € 13.61 + € 99.34 = € 127.23$$

Sum up and conclusion

- ❑ A dollar today is worth more than a dollar tomorrow.
- ❑ Time value of money is expressed concretely as an interest rate.
- ❑ Interest is the price paid for borrowing money.
- ❑ Interest rate it is interest expressed as percentage of the principal.
- ❑ Present Value is the discounted magnitude of a cash flow available at a future date.
- ❑ Future Value is the capitalized magnitude of a cash flow available at a present date.
- ❑ Cash flow are the amounts of money that will flow to and from an investor over time.
- ❑ Cash flow stream is a series of cash flows over several periods.