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QMEFM - LECTURE 3 (APPENDIX)

$$y_t = \beta_1 + \beta_2 X_t + u_t \qquad t = 1 \ldots T$$

OLS PROBLEM: $\displaystyle \min_{\hat\beta_1, \hat\beta_2} \sum_{t=1}^{T} \hat u_t^2 ,$

WHERE $\hat u_t = y_t - (\hat\beta_1 + \hat\beta_2 X_t)$ (RESIDUALS)

$$\sum_t \hat{u}_t^2 = \sum_t \left(y_t - \hat{\beta}_1 - \hat{\beta}_2 x_t\right)^2 =$$

$$= \sum_t y_t^2 + \sum_t \hat{\beta}_1^2 + \hat{\beta}_2^2 \sum_t x_t^2 - 2\hat{\beta}_1 \sum_t y_t$$

$$- 2\hat{\beta}_2 \sum_t y_t x_t + 2\hat{\beta}_1 \hat{\beta}_2 \sum_t x_t$$

FOC: $\dfrac{\partial \sum_t \hat{u}_t^2}{\partial \hat{\beta}_1} = 0$; $\dfrac{\partial \sum_t \hat{u}_t^2}{\partial \hat{\beta}_2} = 0$

$$\frac{\partial \sum_t \hat{u}_t^2}{\partial \hat{\beta}_1} = 2T\hat{\beta}_1 - 2\sum_t y_t + 2\hat{\beta}_2 \sum_t X_t$$

SINCE $\sum_t \hat{\beta}_1^2 = T\hat{\beta}_1^2$

EQUATING TO ZERO: $\cancel{2}T\hat{\beta}_1 - \cancel{2}\sum_t y_t + \cancel{2}\hat{\beta}_2 \sum_t X_t = 0$

$$T\hat{\beta}_1 - T\bar{y} + \hat{\beta}_2 T \bar{X} = 0$$

SINCE $\bar{y} = \frac{\sum_t y_t}{T}$ AND $\bar{X} = \frac{\sum_t X_t}{T}$

HENCE : $\hat{\beta}_1 - \bar{y} + \hat{\beta}_2 \bar{x} = 0$

$$\boxed{\hat{\beta}_1 = \bar{y} - \hat{\beta}_2 \bar{x}}$$

$$\frac{\partial \sum_t \hat{u}_t^2}{\partial \hat{\beta}_2} = 2\hat{\beta}_2 \sum_t X_t^2 - 2\sum_t Y_t X_t + 2\hat{\beta}_1 \sum_t X_t$$

$$= 2\hat{\beta}_2 \sum_t X_t^2 - 2\sum_t Y_t X_t + 2\left(\bar{Y} - \hat{\beta}_2 \bar{X}\right)\sum_t X_t$$

AFTER SUBSTITUTING FOR $\hat{\beta}_1$

$$= 2\hat{\beta}_2 \sum_t x_t^2 - 2\sum_t y_t x_t + 2\bar{y}\sum_t x_t - 2\hat{\beta}_2 \bar{x} \sum_t x_t$$

$$= 2\hat{\beta}_2 \sum_t x_t^2 - 2\sum_t y_t x_t + 2T\bar{y}\bar{x} - 2\hat{\beta}_2 T\bar{x}^2$$

EQUATING TO ZERO :

$$\cancel{2}\hat{\beta}_2 \sum_t x_t^2 - \cancel{2}\sum_t y_t x_t + \cancel{2}T\bar{y}\bar{x} - \cancel{2}\hat{\beta}_2 T\bar{x}^2 = 0$$

$$\hat{\beta}_2\left(\sum_t x_t^2 - T\bar{x}^2\right) = \sum_t y_t x_t - T\bar{y}\bar{x}$$

$$\hat{\beta}_2 = \frac{\sum_t y_t x_t - T\bar{y}\bar{x}}{\sum_t x_t^2 - T\bar{x}^2}$$

N.B. $\sum_t (x_t - \bar{x})^2 = \sum_t x_t^2 + \sum_t \bar{x}^2 - 2\bar{x}\sum_t x_t =$

$= \sum_t x_t^2 + T\bar{x}^2 - 2T\bar{x}^2 = \sum_t x_t^2 - T\bar{x}^2$

$$\underline{x}\,\underline{y}\,\underline{1} - \tau x\,\tau y\,\tau \underline{2} = \cancel{\underline{x}\,\underline{y}\,\underline{1}} +$$

$$+\ \underline{y}\,\underline{x}\,\underline{1} - \cancel{\underline{x}\,\underline{y}\,\underline{1}} - \tau x\,\tau y\,\tau \underline{3} =$$

$$= \underline{x}\,\underline{y}\,\tau\underline{2} +$$

$$\tau y\,\tau \underline{3}\,\underline{x} - \tau x\,\tau \underline{3}\,\underline{y} - \tau x\,\tau y\,\tau \underline{3} = (\underline{x} - \tau x)(\underline{y} - \tau y)\,\tau \underline{2}$$

HENCE :

$$\hat{\beta}_2 = \frac{\sum_t y_t x_t - T\bar{x}\bar{y}}{\sum_t x_t^2 - T\bar{x}^2} = \boxed{\frac{\sum_t (y_t - \bar{y})(x_t - \bar{x})}{\sum_t (x_t - \bar{x})^2}}$$

SOC : $\dfrac{\partial^2 \sum_t \hat{u}_t^2}{\partial \hat{\beta}_1^2}$ HAS TO BE > 0 FOR A MINIMUM

$\dfrac{\partial^2 \sum_t \hat{u}_t^2}{\partial \hat{\beta}_2^2}$ HAS TO BE > 0 FOR A MINIMUM

$$\frac{\partial^2 \sum_t \hat{u}_t^2}{\partial \hat{\beta}_1^2} = \frac{\partial}{\partial \hat{\beta}_1^2}\left(T\hat{\beta}_1 - \sum_t y_t + \hat{\beta}_2 \sum_t x_t\right)$$

$$= T > 0$$

$$\frac{\partial^2 \sum_t \hat{u}_t^2}{\partial \hat{\beta}_2} = \frac{\partial}{\partial \hat{\beta}_2^2}\left(\hat{\beta}_2 \sum_t x_t^2 - \sum_t y_t x_t + T\bar{y}\bar{x} - T\hat{\beta}_2 \bar{x}^2\right) =$$

$$= \sum_t x_t^2 - T\bar{x}^2 = \sum_t (x_t - \bar{x})^2 > 0$$