

# ● “Metodi Matematici per le applicazioni industriali”



Facoltà di Ingegneria

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## **ANALYTIC NETWORK PROCESS**

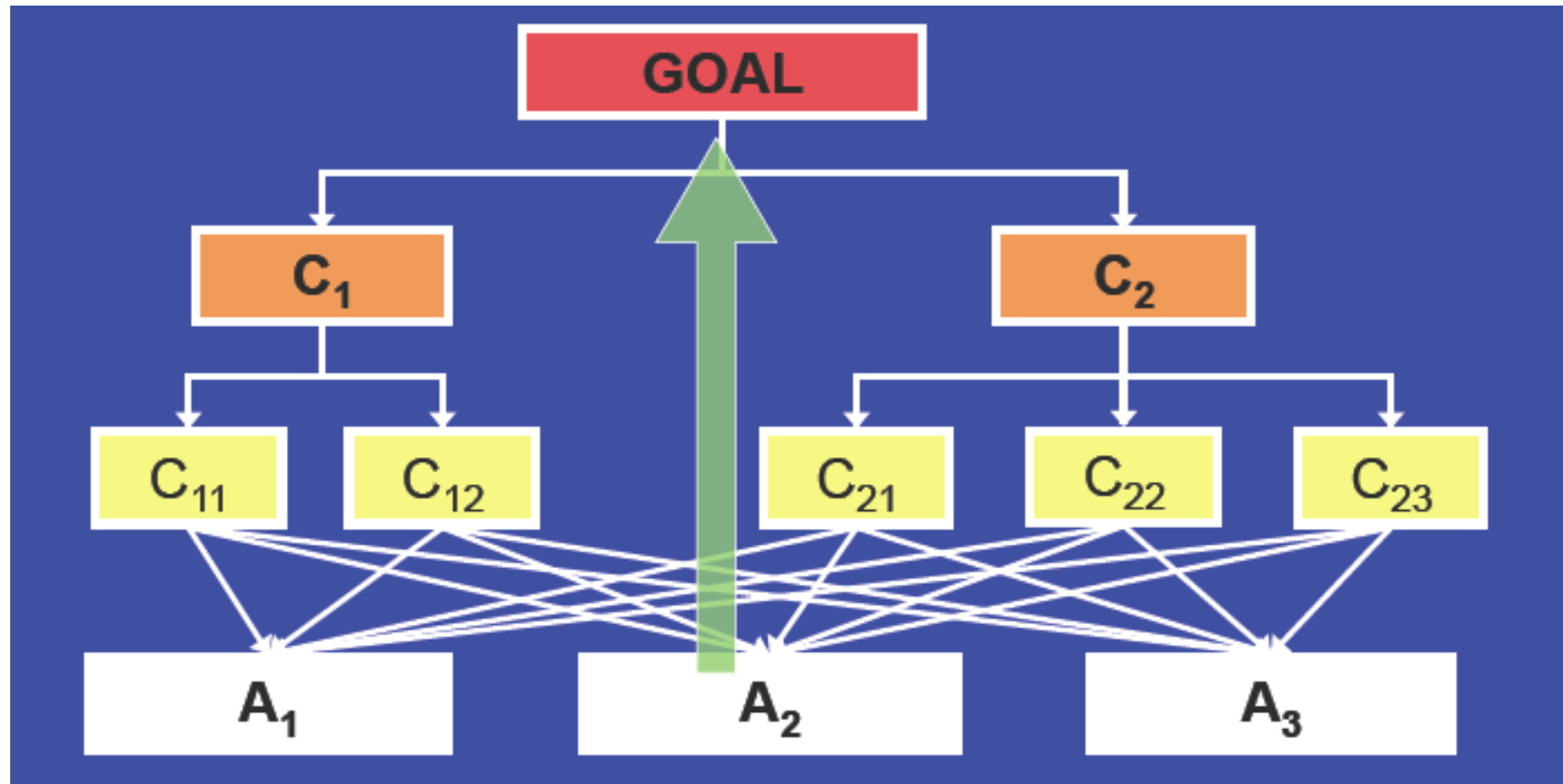
### **Decision making with dependence and feedback**

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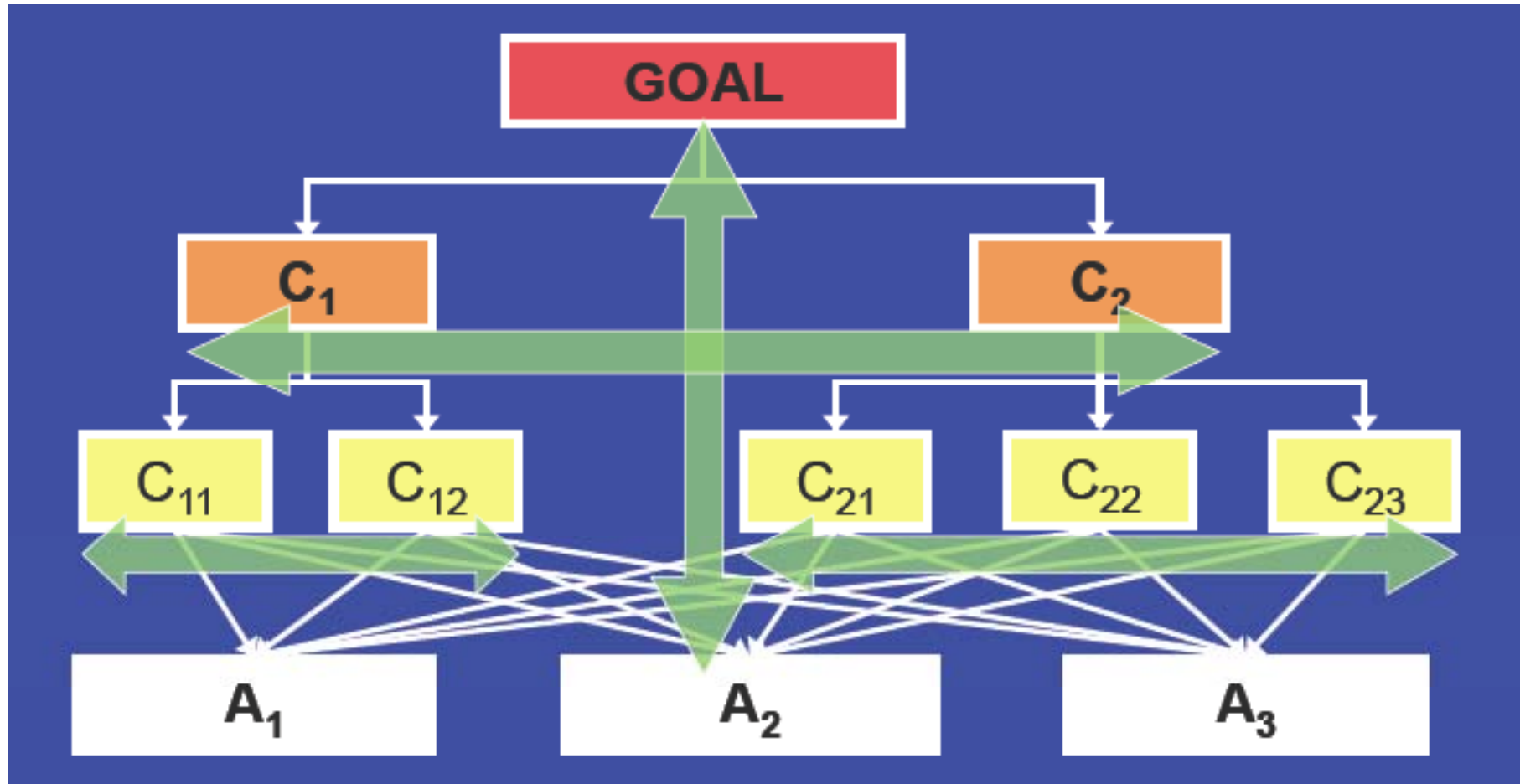
## ● Introduction

- ✓ Profesor Thomas L. Saaty 90's
  - ✓ Generalization of AHP.
  - ✓ ANP allows to include relations of dependence and feedback among elements of the system.
  - ✓ The problem modelization is more complex and more realistic.
- 
- ✓ Thomas L. Saaty "Decision making with independence and feedback: The Analytic Network Process" RWS Publications, Pittsburgh (2001)

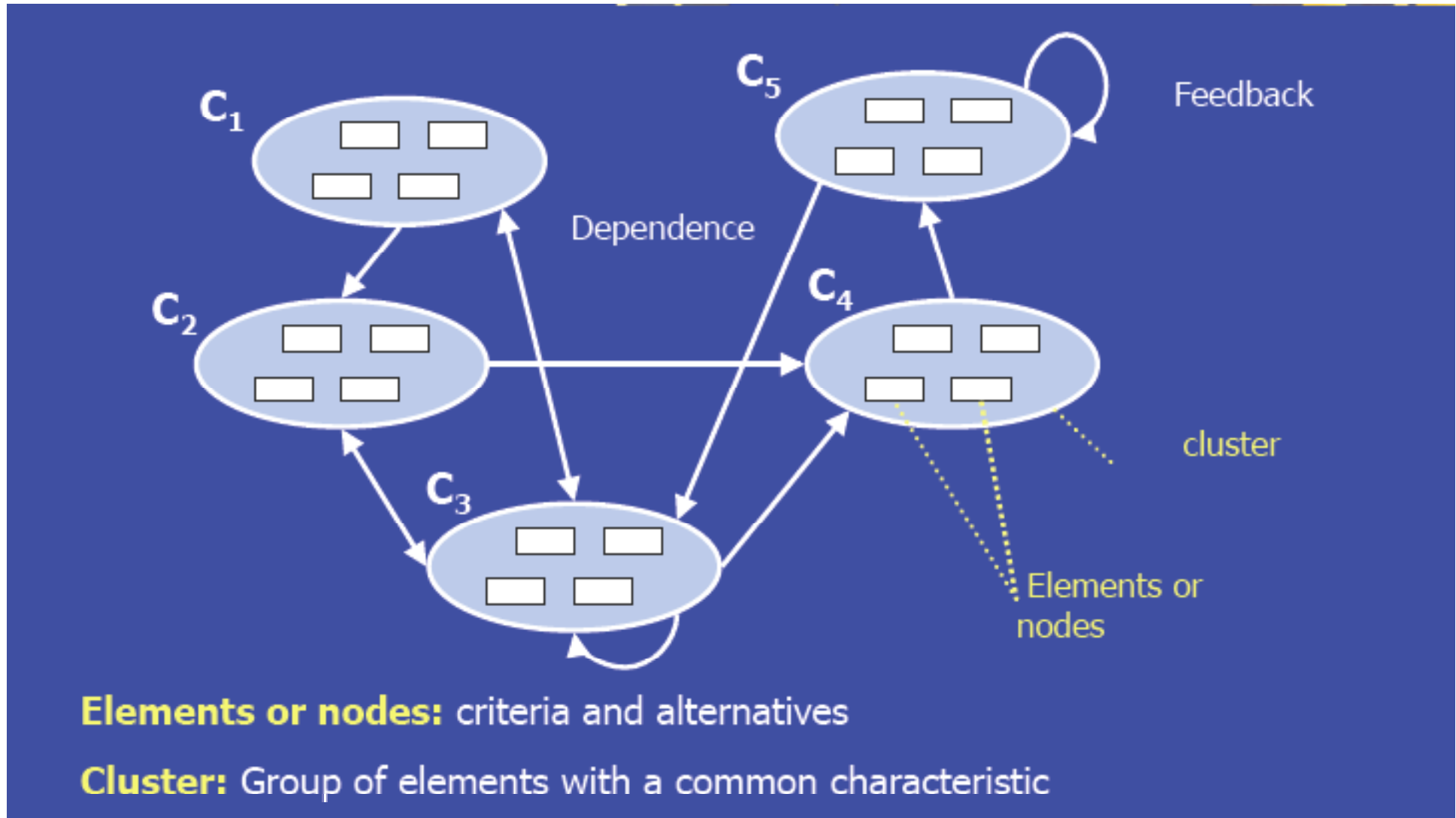
## ● Analytic Hierarchy Process: One-direction influence



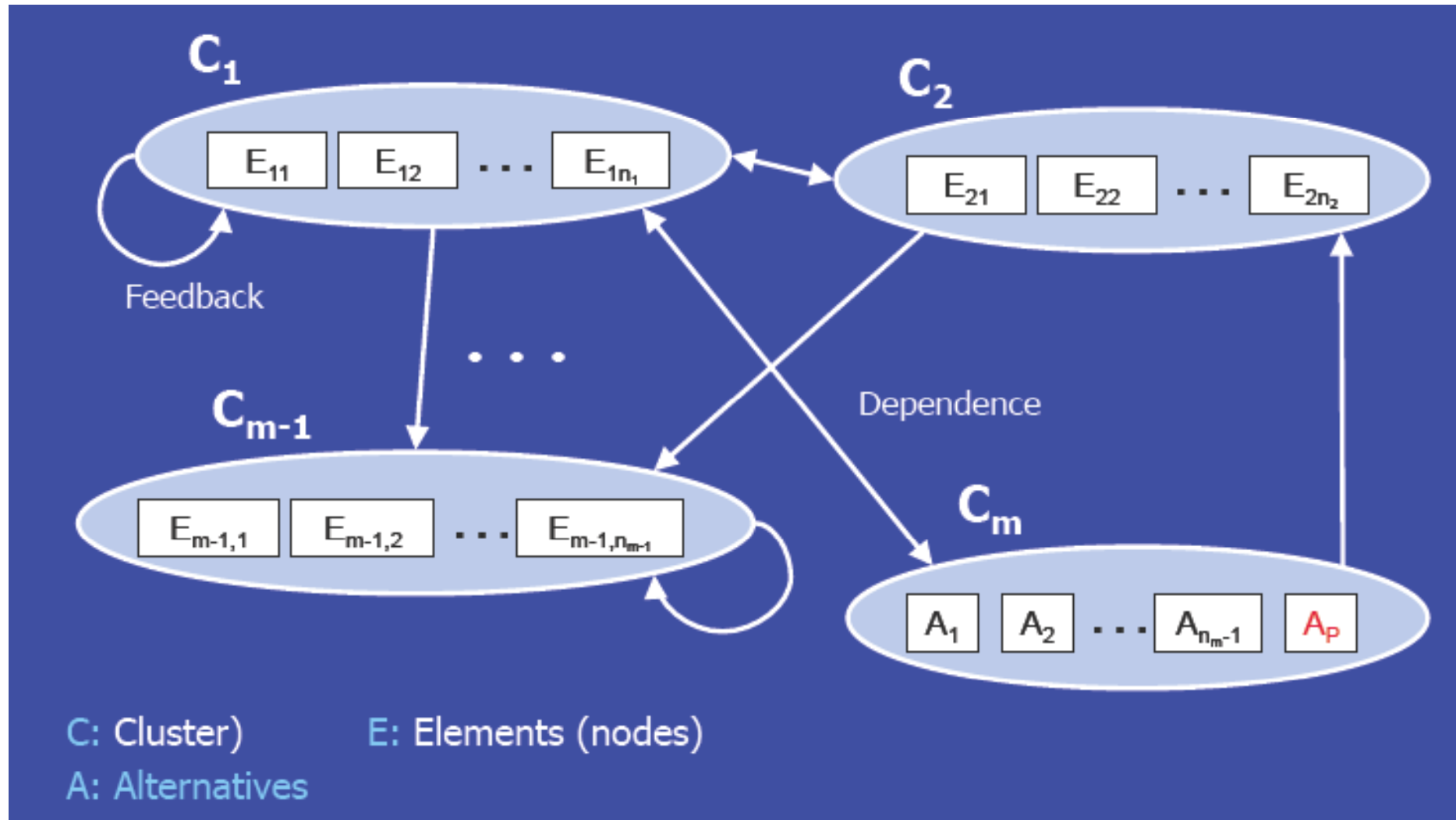
## ● Analytic Network Process (ANP): Interdependence and feedback



## ● Analytic Network Process (ANP): Network model



## Network model



## ● Inner and Outer Dependence

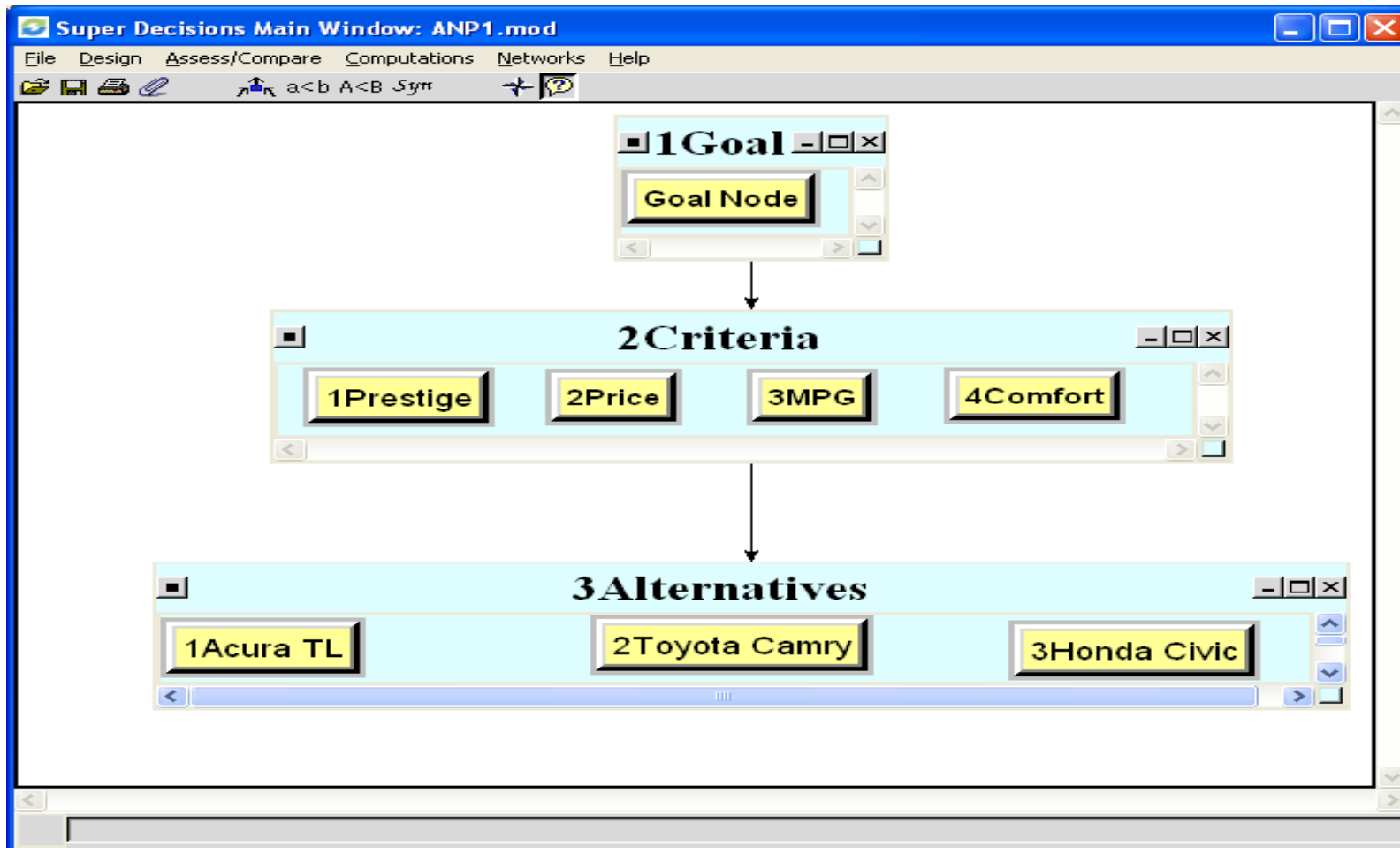
### **Outer dependence**

The parent node and the nodes to be compared are in different clusters. A directed link appears *from* the parent node cluster to the other cluster.

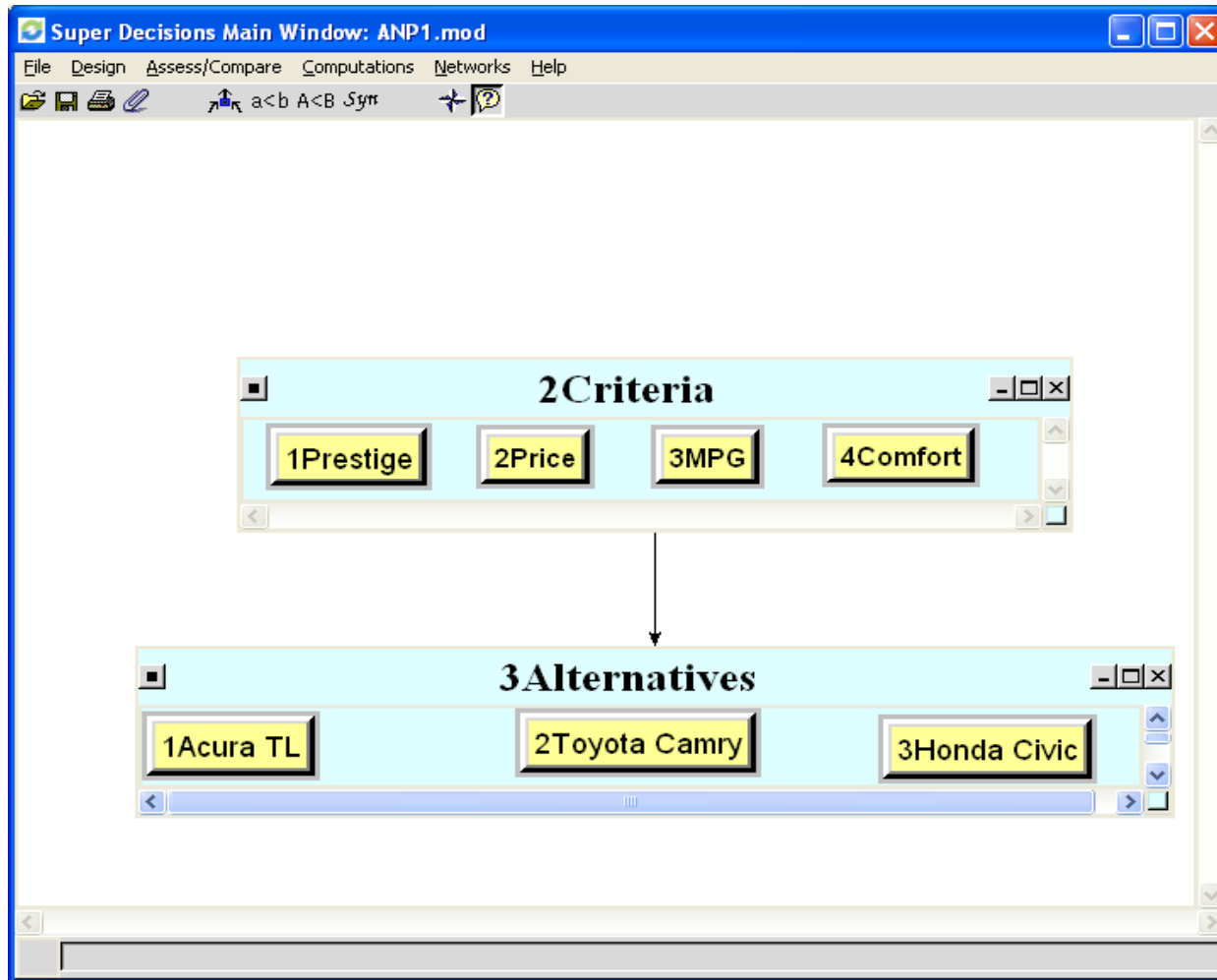
### **Inner dependence**

The parent node and the nodes to be compared are in the same cluster. The cluster is linked to itself and a loop link appears.

## ● Example: Changing from a Hierarchy to a Network Model



## ● Network Models do not have goals



A network model has a criteria cluster and an alternatives cluster, but no goal.

## ● Evaluation methodology based on ANP

1. MODELIZATION OF THE PROBLEM AS A NETWORK
2. WEIGHTING OF THE ELEMENTS BY MEANS OF ANP

## ● MODELIZATION OF THE PROBLEM AS A NETWORK

- i. Identification of the elements of the network (criteria and alternatives)
- ii. Grouping of the elements into clusters
- iii. Analysis of the influences within the network

## ● Influence matrix

		$C_1$				$C_2$				...	$C_m$			
		$e_{11}$	$e_{12}$	...	$e_{1n_1}$	$e_{21}$	$e_{22}$	...	$e_{2n_2}$	...	$e_{m1}$	$e_{m2}$	...	$e_{mn_m}$
$C_1$	$e_{11}$	$A_{11}$				$A_{12}$				...	$A_{1m}$			
	$e_{12}$													
	...													
	$e_{1n_1}$													
$C_2$	$e_{21}$	$A_{21}$				$A_{22}$				...	$A_{2m}$			
	$e_{22}$													
	...													
	$e_{2n_2}$													
...	...				...				...	...				
$C_m$	$e_{m1}$	$A_{m1}$				$A_{m2}$				...	$A_{mm}$			
	$e_{m2}$													
	...													
	$e_{mn_m}$													

## ● Influence matrix

### Block $A_{ij}$ of the matrix

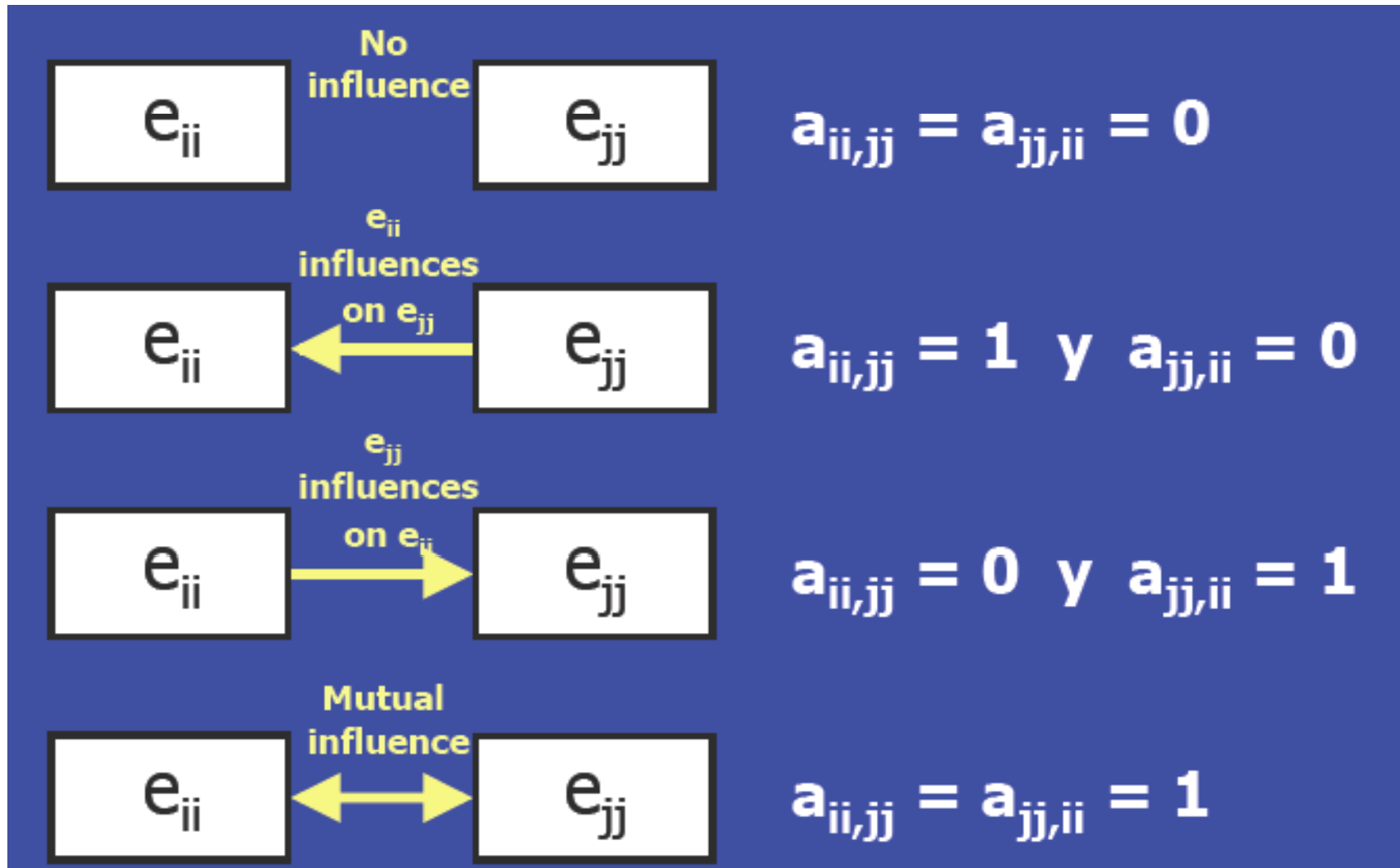
		$C_j$			
		$e_{j1}$	$e_{j2}$	...	$e_{jn_j}$
$C_i$	$e_{i1}$	$a_{i1,j1}$	$a_{i1,j2}$	...	$a_{i1,jn_j}$
	$e_{i2}$	$a_{i2,j1}$	$a_{i2,j2}$	...	$a_{i2,jn_j}$
	...	...	...	...	...
	$e_{in_i}$	$a_{in_i,j1}$	$a_{in_i,j2}$	...	$a_{in_i,jn_j}$

$a_{ii,jj}$ : influence of the elements  $e_{ii}$  on the elements  $e_{jj}$

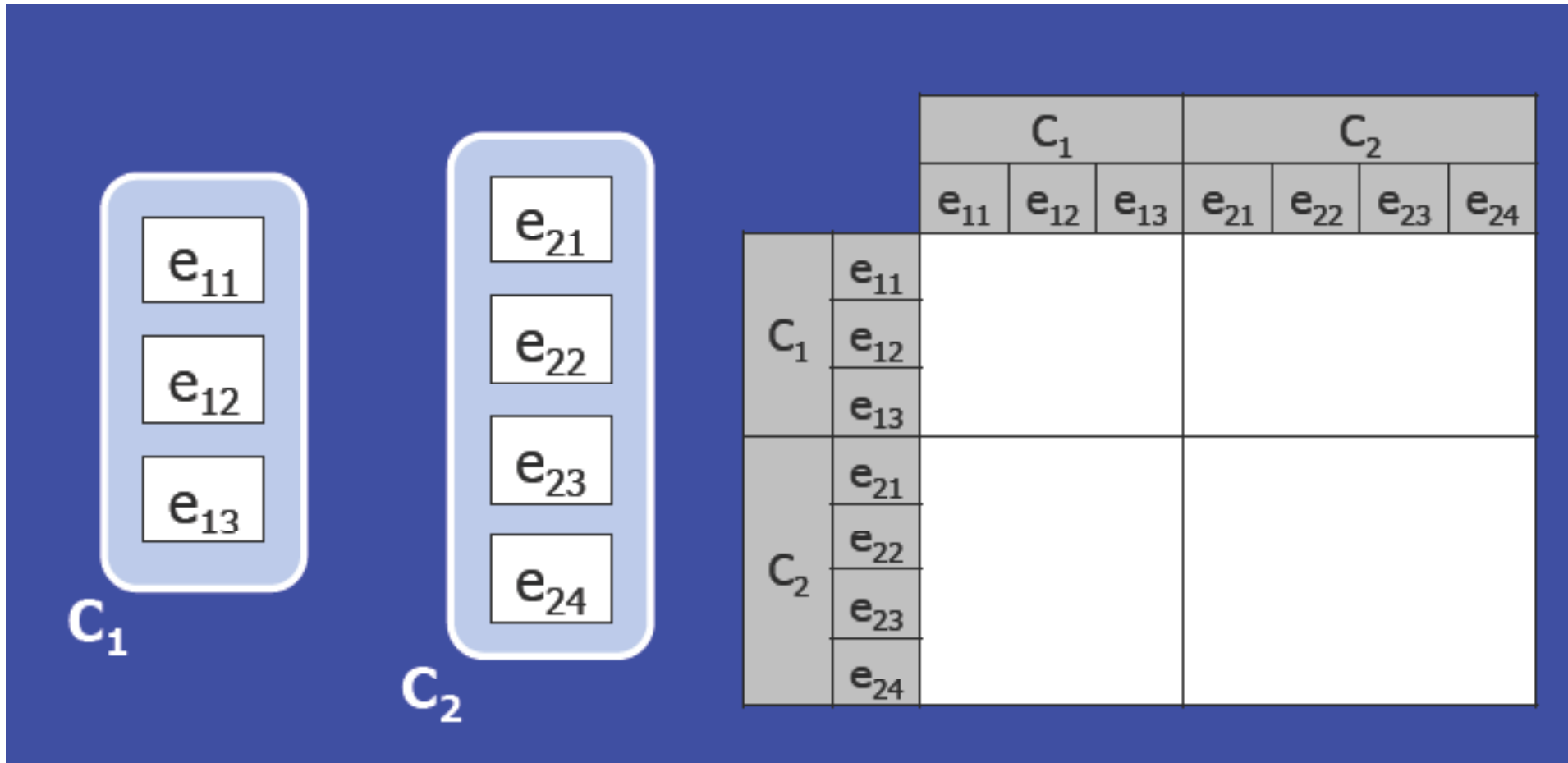
$a_{ii,jj} = 1 \rightarrow e_{ii}$  influences  $e_{jj}$

$a_{ii,jj} = 0 \rightarrow e_{ii}$  does not influence  $e_{jj}$

## ● Influence matrix

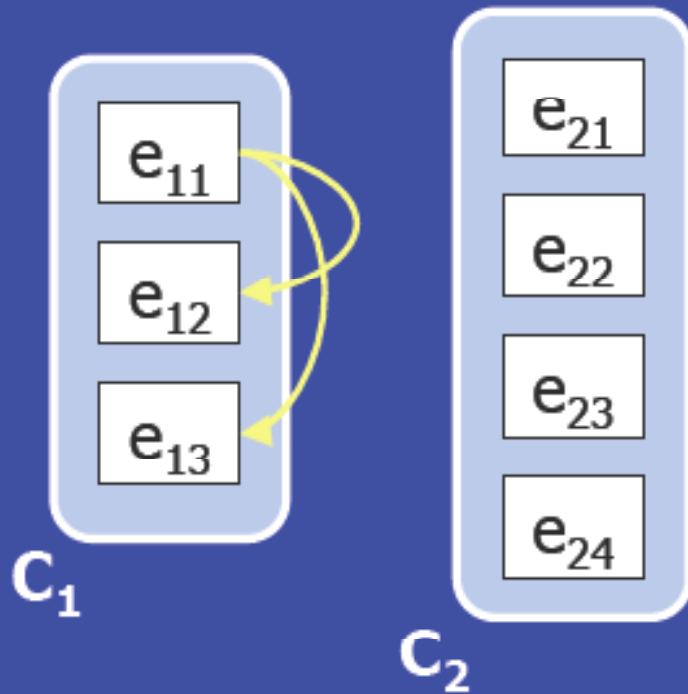


## ● Influence matrix - example



## ● Influence matrix - example

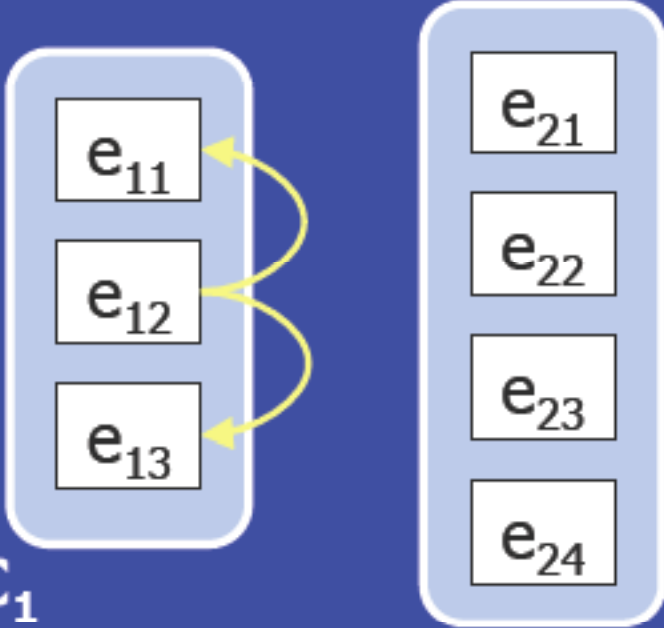
Which elements of  $C_1$  influence on  $e_{11}$ ?



		$C_1$			$C_2$			
		$e_{11}$	$e_{12}$	$e_{13}$	$e_{21}$	$e_{22}$	$e_{23}$	$e_{24}$
$C_1$	$e_{11}$	0						
	$e_{12}$	1						
	$e_{13}$	1						
$C_2$	$e_{21}$							
	$e_{22}$							
	$e_{23}$							
	$e_{24}$							

## ● Influence matrix - example

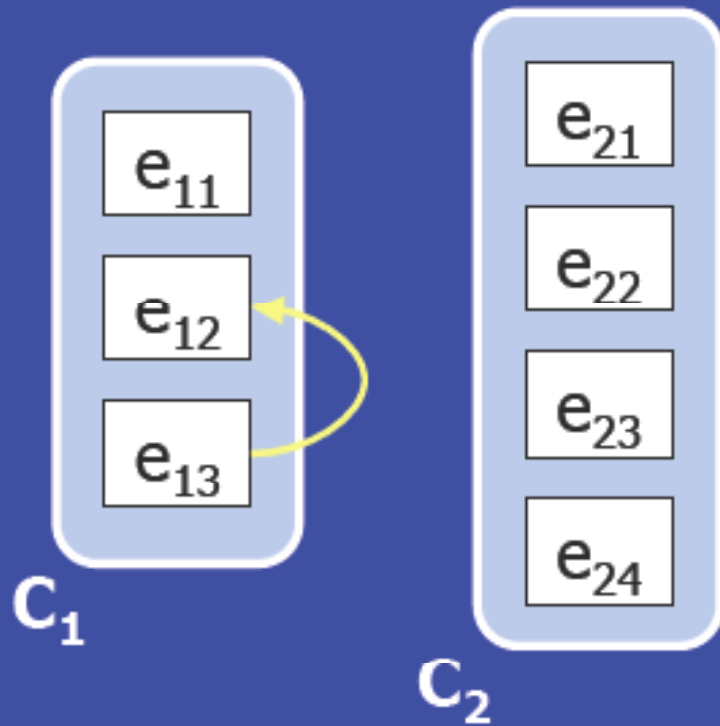
Which elements of  $C_1$  influence on  $e_{12}$ ?



		$C_1$			$C_2$			
		$e_{11}$	$e_{12}$	$e_{13}$	$e_{21}$	$e_{22}$	$e_{23}$	$e_{24}$
$C_1$	$e_{11}$	0	1					
	$e_{12}$	1	0					
	$e_{13}$	1	1					
$C_2$	$e_{21}$							
	$e_{22}$							
	$e_{23}$							
	$e_{24}$							

## ● Influence matrix - example

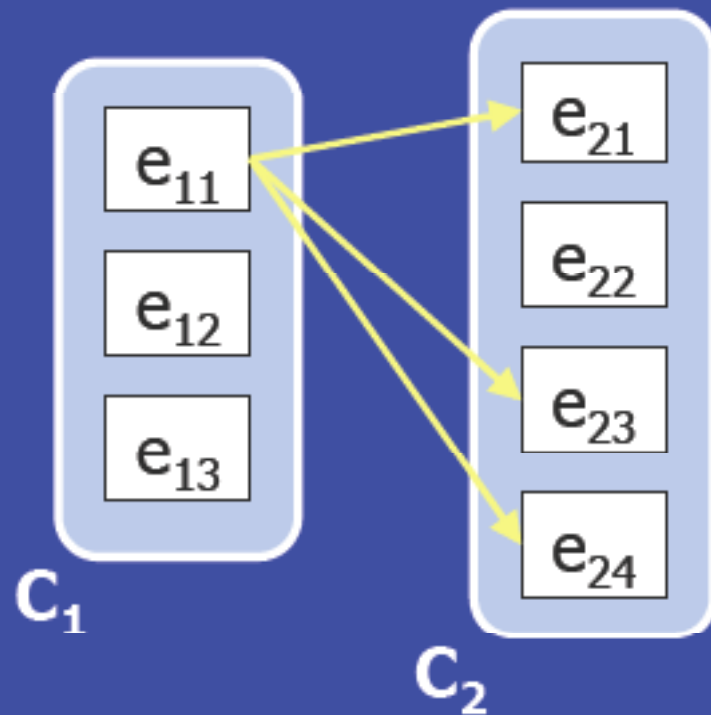
Which elements of  $C_1$  influence on  $e_{13}$ ?



		$C_1$			$C_2$			
		$e_{11}$	$e_{12}$	$e_{13}$	$e_{21}$	$e_{22}$	$e_{23}$	$e_{24}$
$C_1$	$e_{11}$	0	1	0				
	$e_{12}$	1	0	1				
	$e_{13}$	1	1	0				
$C_2$	$e_{21}$							
	$e_{22}$							
	$e_{23}$							
	$e_{24}$							

## ● Influence matrix - example

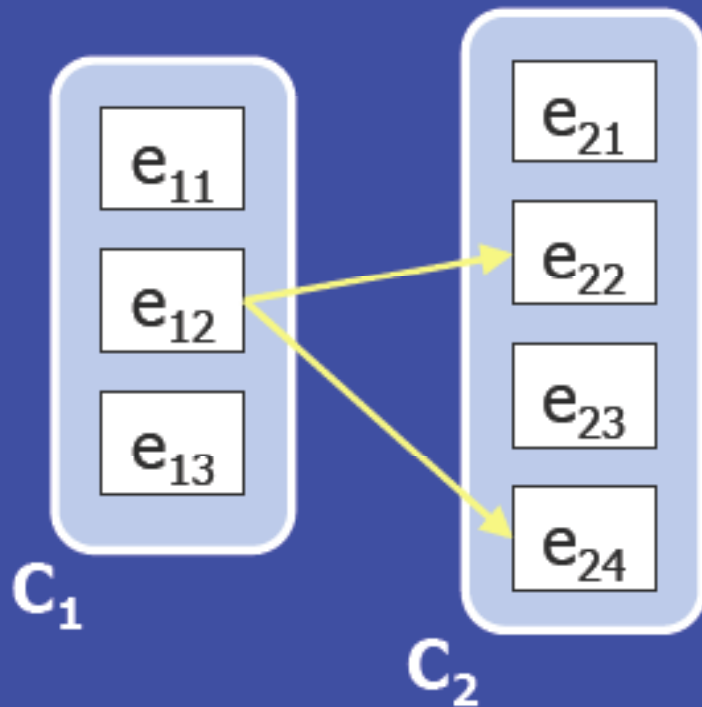
Which elements of  $C_2$  influence on  $e_{11}$ ?



		$C_1$			$C_2$			
		$e_{11}$	$e_{12}$	$e_{13}$	$e_{21}$	$e_{22}$	$e_{23}$	$e_{24}$
$C_1$	$e_{11}$	0	1	0				
	$e_{12}$	1	0	1				
	$e_{13}$	1	1	0				
$C_2$	$e_{21}$	1						
	$e_{22}$	0						
	$e_{23}$	1						
	$e_{24}$	1						

## ● Influence matrix - example

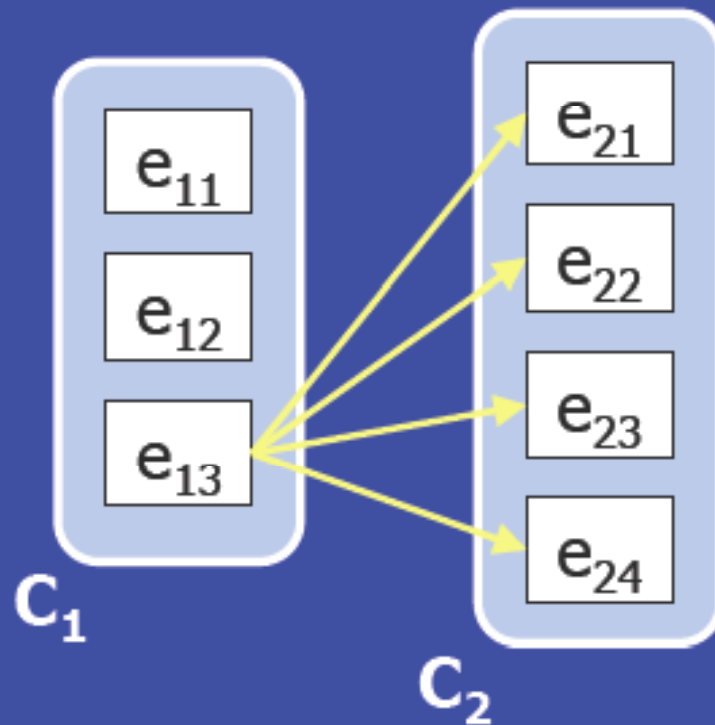
Which elements of  $C_2$  influence on  $e_{12}$ ?



		$C_1$			$C_2$			
		$e_{11}$	$e_{12}$	$e_{13}$	$e_{21}$	$e_{22}$	$e_{23}$	$e_{24}$
$C_1$	$e_{11}$	0	1	0				
	$e_{12}$	1	0	1				
	$e_{13}$	1	1	0				
$C_2$	$e_{21}$	1	0					
	$e_{22}$	0	1					
	$e_{23}$	1	0					
	$e_{24}$	1	1					

## ● Influence matrix - example

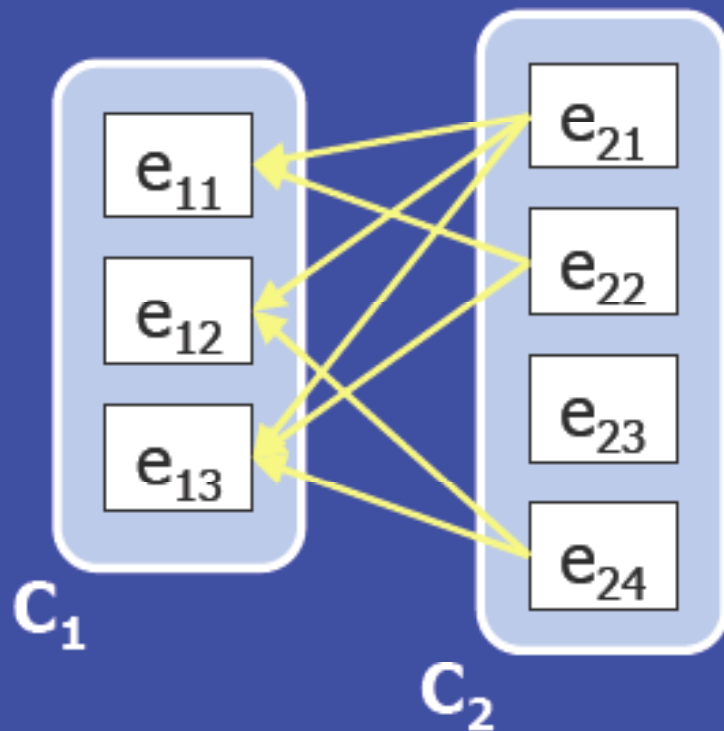
Which elements of  $C_2$  influence on  $e_{13}$ ?



		$C_1$			$C_2$			
		$e_{11}$	$e_{12}$	$e_{13}$	$e_{21}$	$e_{22}$	$e_{23}$	$e_{24}$
$C_1$	$e_{11}$	0	1	0				
	$e_{12}$	1	0	1				
	$e_{13}$	1	1	0				
$C_2$	$e_{21}$	1	0	1				
	$e_{22}$	0	1	1				
	$e_{23}$	1	0	1				
	$e_{24}$	1	1	1				

## ● Influence matrix - example

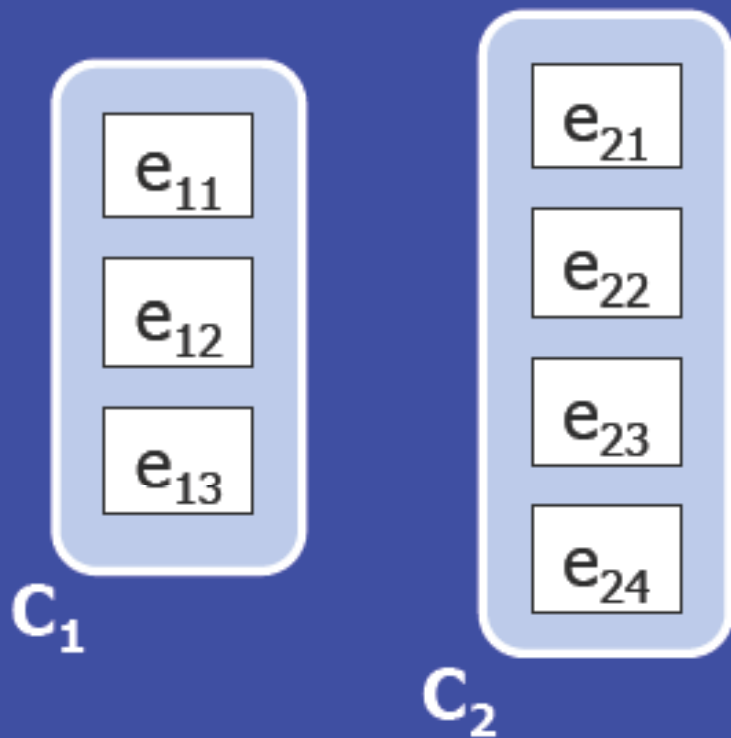
Which elements of  $C_1$  influence on the elements of  $C_2$ ?



		$C_1$			$C_2$			
		$e_{11}$	$e_{12}$	$e_{13}$	$e_{21}$	$e_{22}$	$e_{23}$	$e_{24}$
$C_1$	$e_{11}$	0	1	0	1	1	0	0
	$e_{12}$	1	0	1	1	0	0	1
	$e_{13}$	1	1	0	1	1	0	1
$C_2$	$e_{21}$	1	0	1				
	$e_{22}$	0	1	1				
	$e_{23}$	1	0	1				
	$e_{24}$	1	1	1				

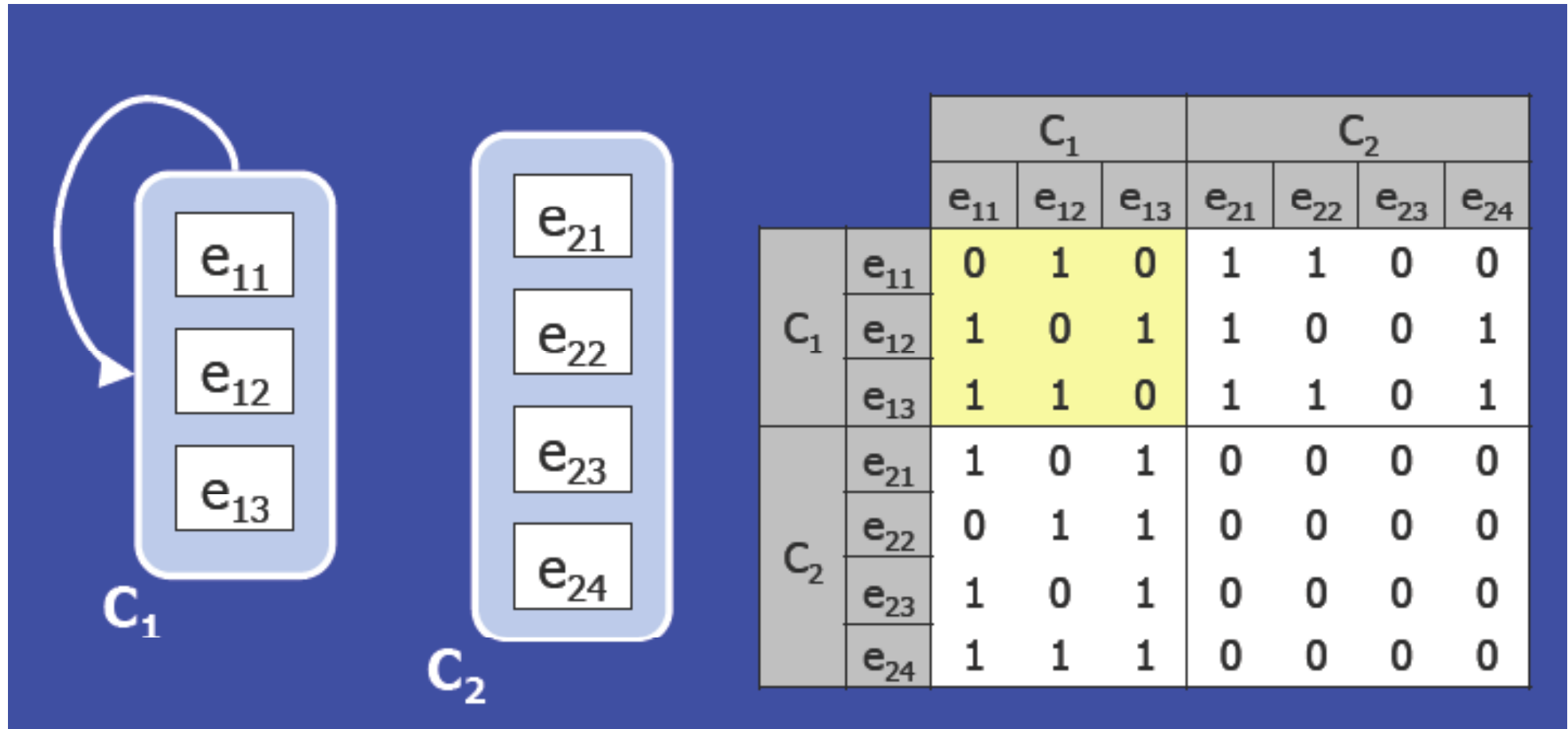
## ● Influence matrix - example

Which elements of  $C_2$  influence on the elements of  $C_2$ ?

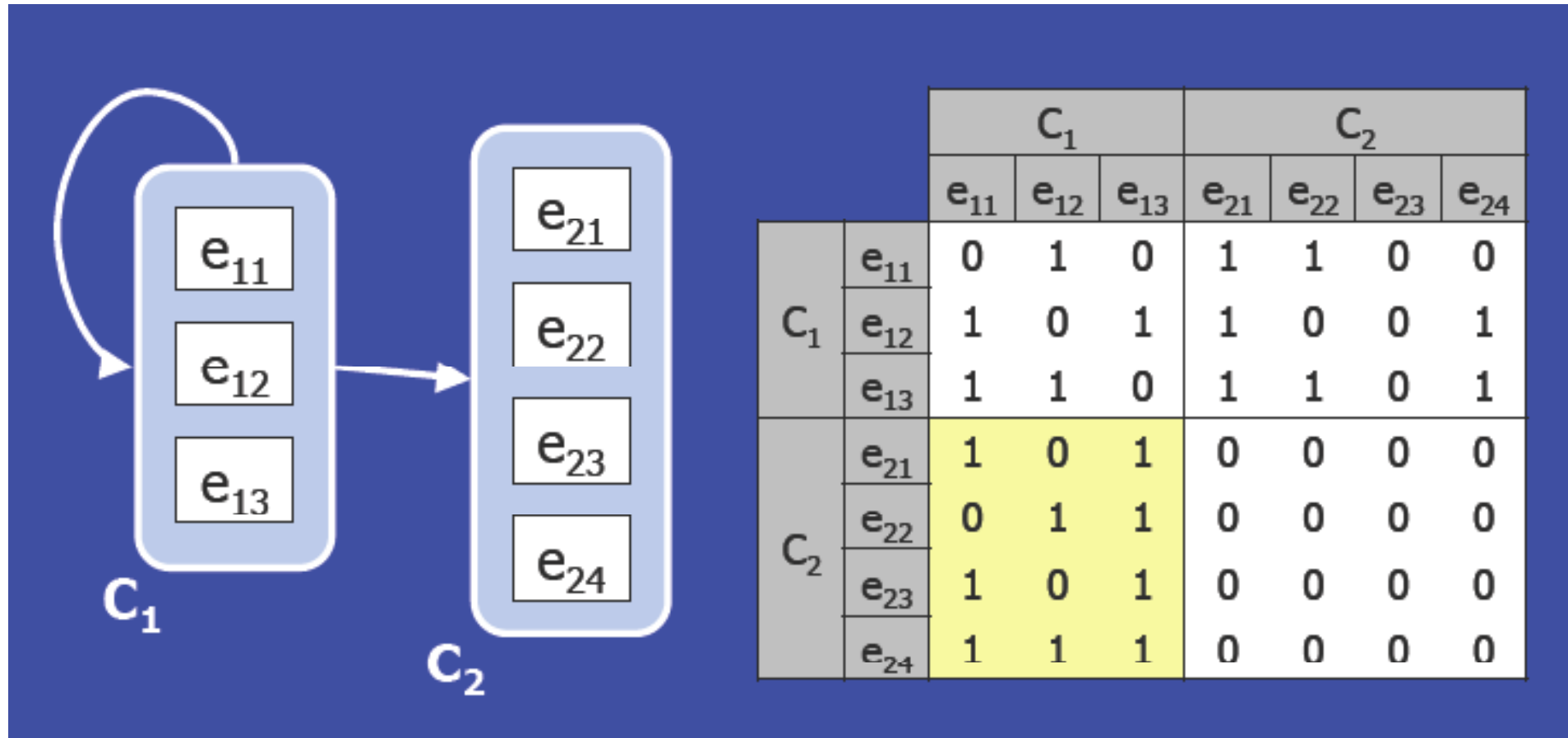


		$C_1$			$C_2$			
		$e_{11}$	$e_{12}$	$e_{13}$	$e_{21}$	$e_{22}$	$e_{23}$	$e_{24}$
$C_1$	$e_{11}$	0	1	0	1	1	0	0
	$e_{12}$	1	0	1	1	0	0	1
	$e_{13}$	1	1	0	1	1	0	1
$C_2$	$e_{21}$	1	0	1	0	0	0	0
	$e_{22}$	0	1	1	0	0	0	0
	$e_{23}$	1	0	1	0	0	0	0
	$e_{24}$	1	1	1	0	0	0	0

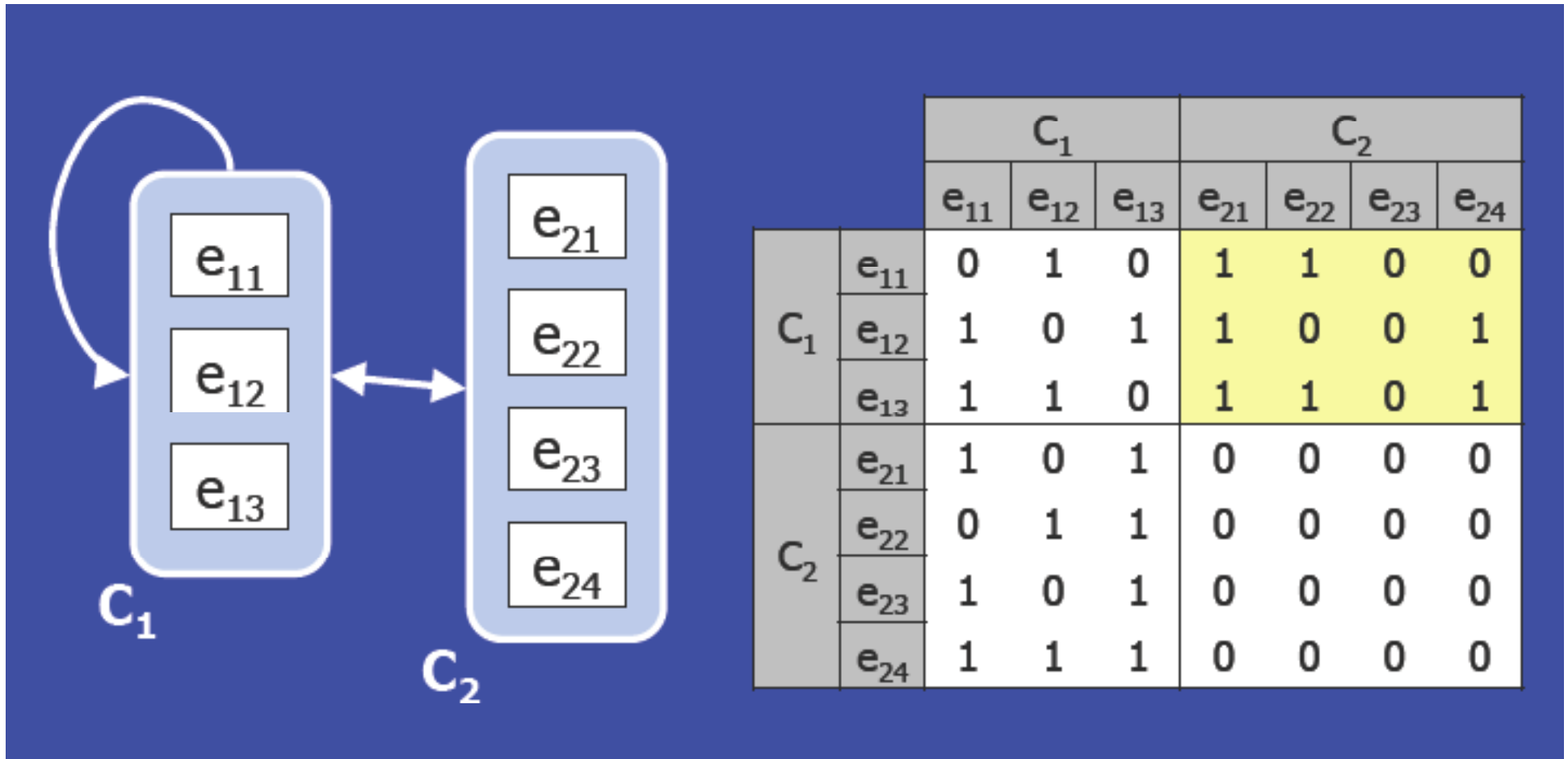
## ● Influence matrix - example



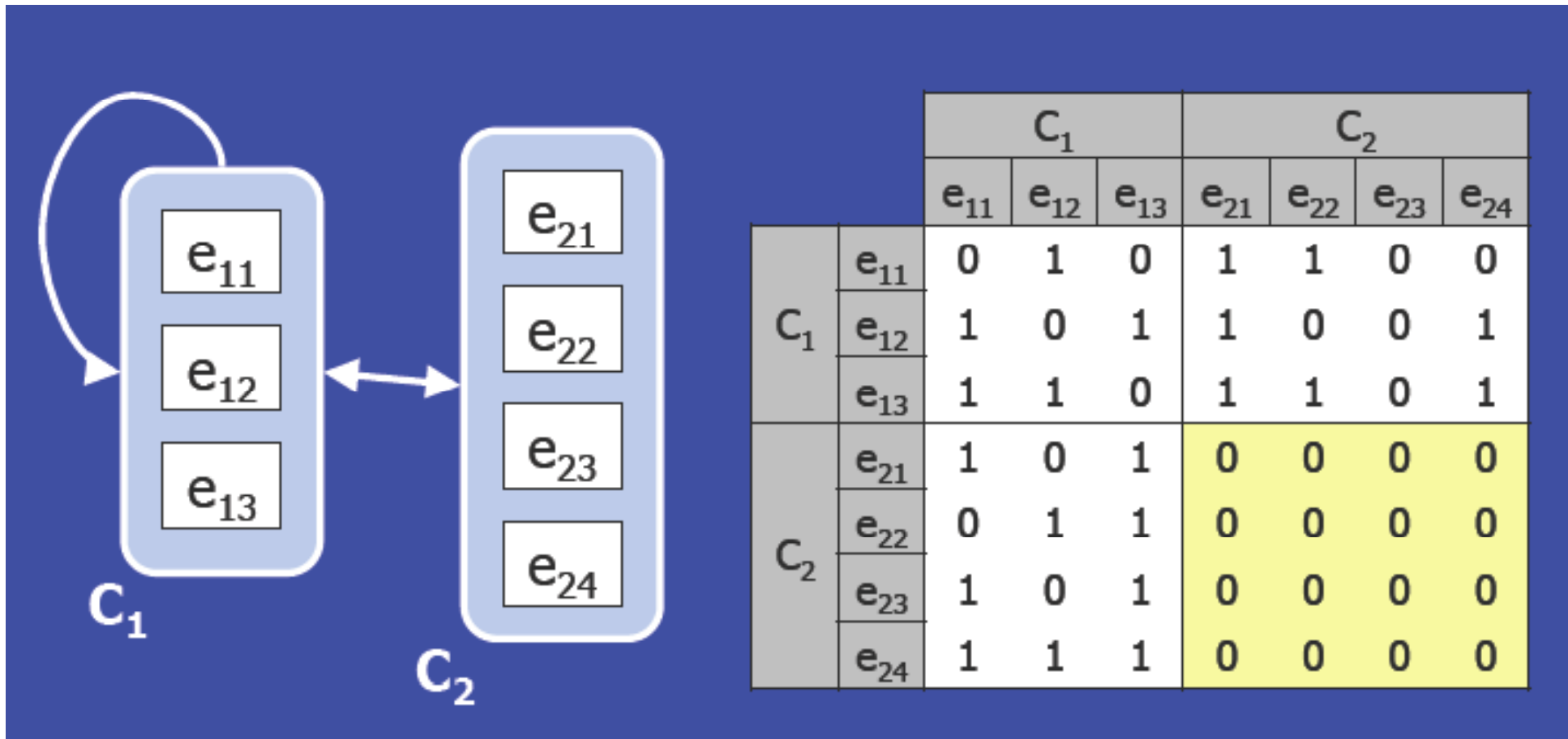
## ● Influence matrix - example



## ● Influence matrix - example



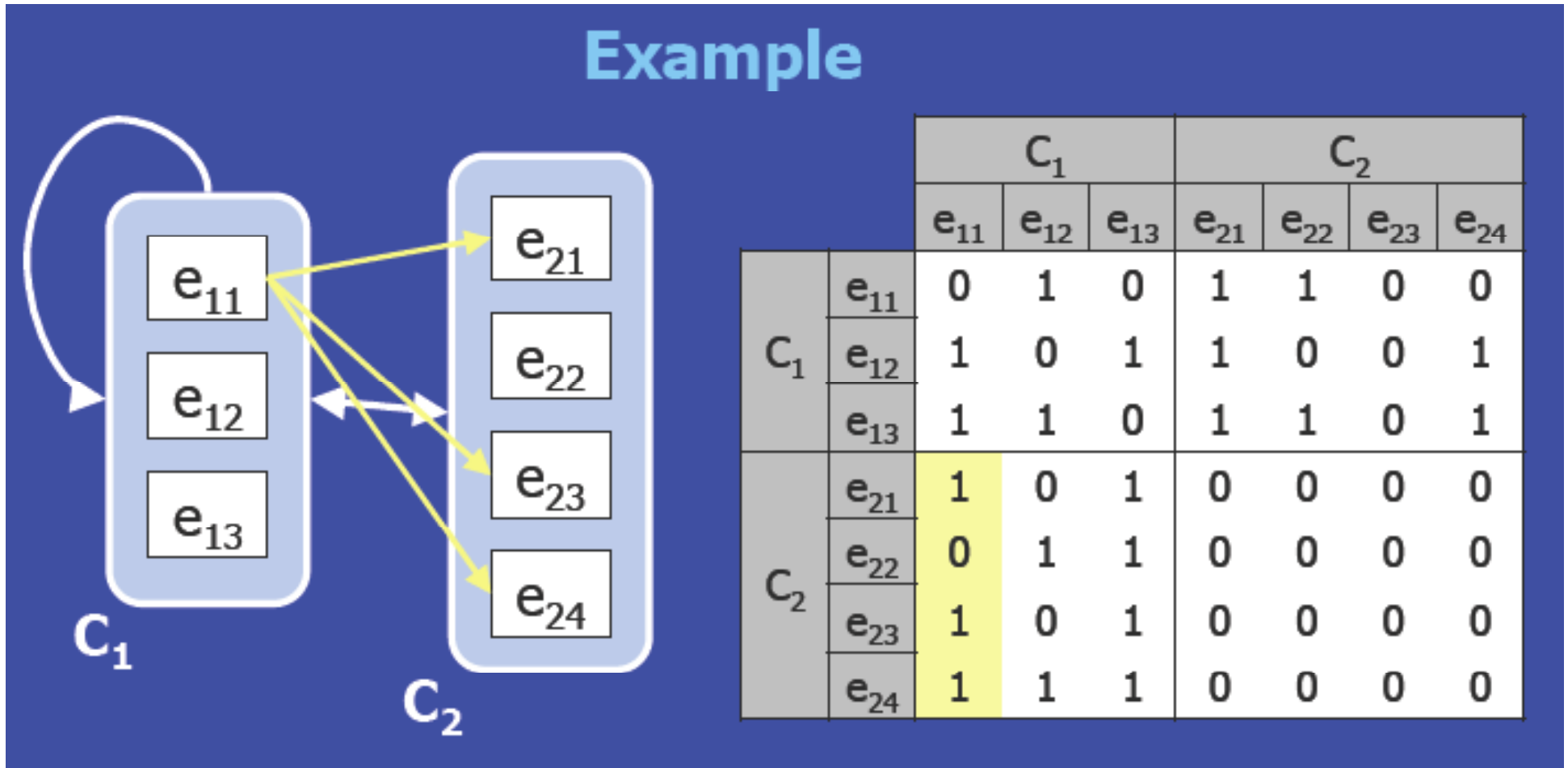
## ● Influence matrix - example



## ● **WEIGHTING OF THE ELEMENTS BY MEANS OF ANP**

- i. Calculate priorities among elements
- ii. Calculate priorities among groups
- iii. Build original supermatrix
- iv. Calculate weighted supermatrix
- v. Calculate limit supermatrix
- vi. Obtain alternatives prioritization

- i. Calculate priorities among elements



- i. Calculate priorities among elements

**Pairwise comparison matrix**

$e_{11}$	$e_{21}$	$e_{23}$	$e_{24}$
$e_{21}$			
$e_{23}$			
$e_{24}$			

**QUESTION:** Given two elements of component C2 which have an influence on the element  $e_{11}$ , How much more does the first element influence on the element  $e_{11}$  than the second element?

**PRIORITIES ARE OBTAINED THE SAME WAY AS FOR AHP.**

- i. Calculate priorities among elements

### Pairwise comparison matrix

$e_{11}$	$e_{21}$	$e_{23}$	$e_{24}$
$e_{21}$	1	$r_{12}$	$r_{13}$
$e_{23}$	$1/r_{12}$	1	$r_{23}$
$e_{24}$	$1/r_{13}$	$1/r_{23}$	1

$$\begin{bmatrix} W_{21,11} \\ W_{23,11} \\ W_{24,11} \end{bmatrix}$$

After consistence is checked, the principal eigenvector becomes the priorities' vector.

- i. Calculate priorities among elements

The priorities vector has the following structure:

$$[w_{21,11} / w_{23,11} / w_{24,11}]^T$$

Before its introduction into the supermatrix this vector has to be completed with 0s:

$$[w_{21,11} / 0 / w_{23,11} / w_{24,11}]^T$$

**SAME PROCEDURE FOR ALL THE ELEMENTS OF THE NETWORK.**

## ● Establishing Priorities for Criteria

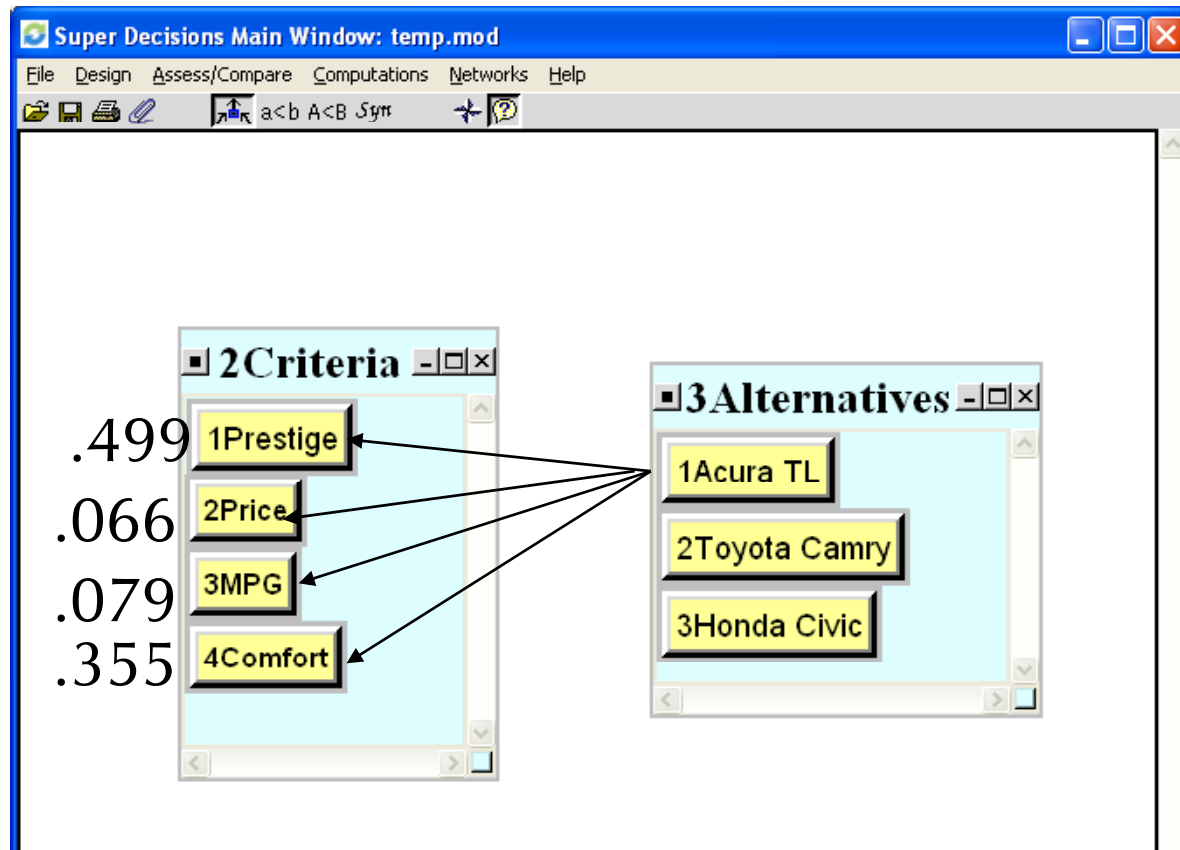
- In AHP one answers the question:

“How important is Criterion1 compared to Criterion2 with respect to the goal?” *Sometimes too abstract a question to answer.*

- In ANP criteria are prioritized by asking how important they are in the alternatives being considered. *Usually an easier question to answer when looking at actual alternatives that have the properties in question.*

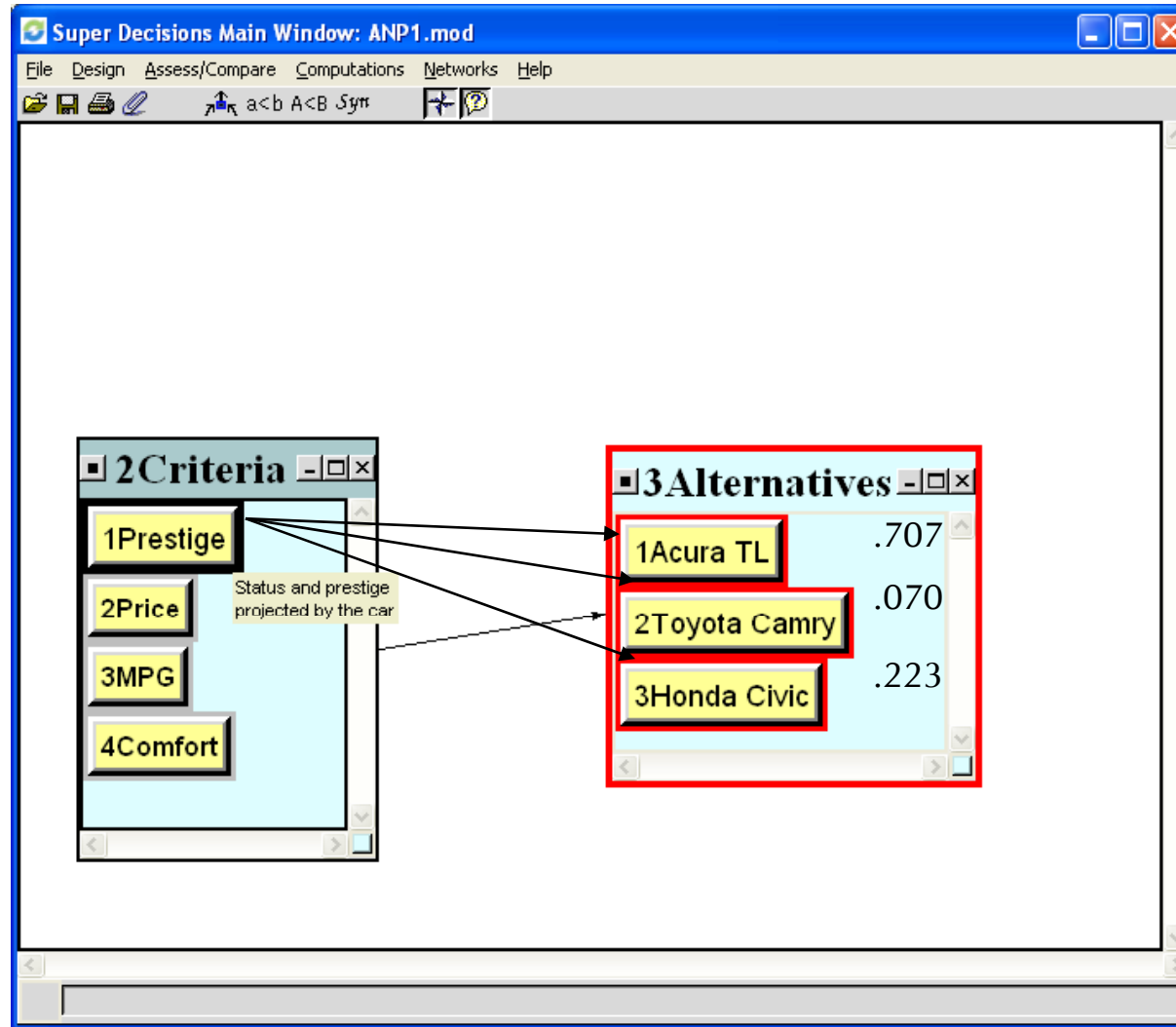
# Feedback: Link Alternatives to Criteria and compare for Preference

For example, a pairwise question would be: “For the Acura TL, which do you like better, its prestige or its price?” ; “Its prestige or its MPG?” etc.

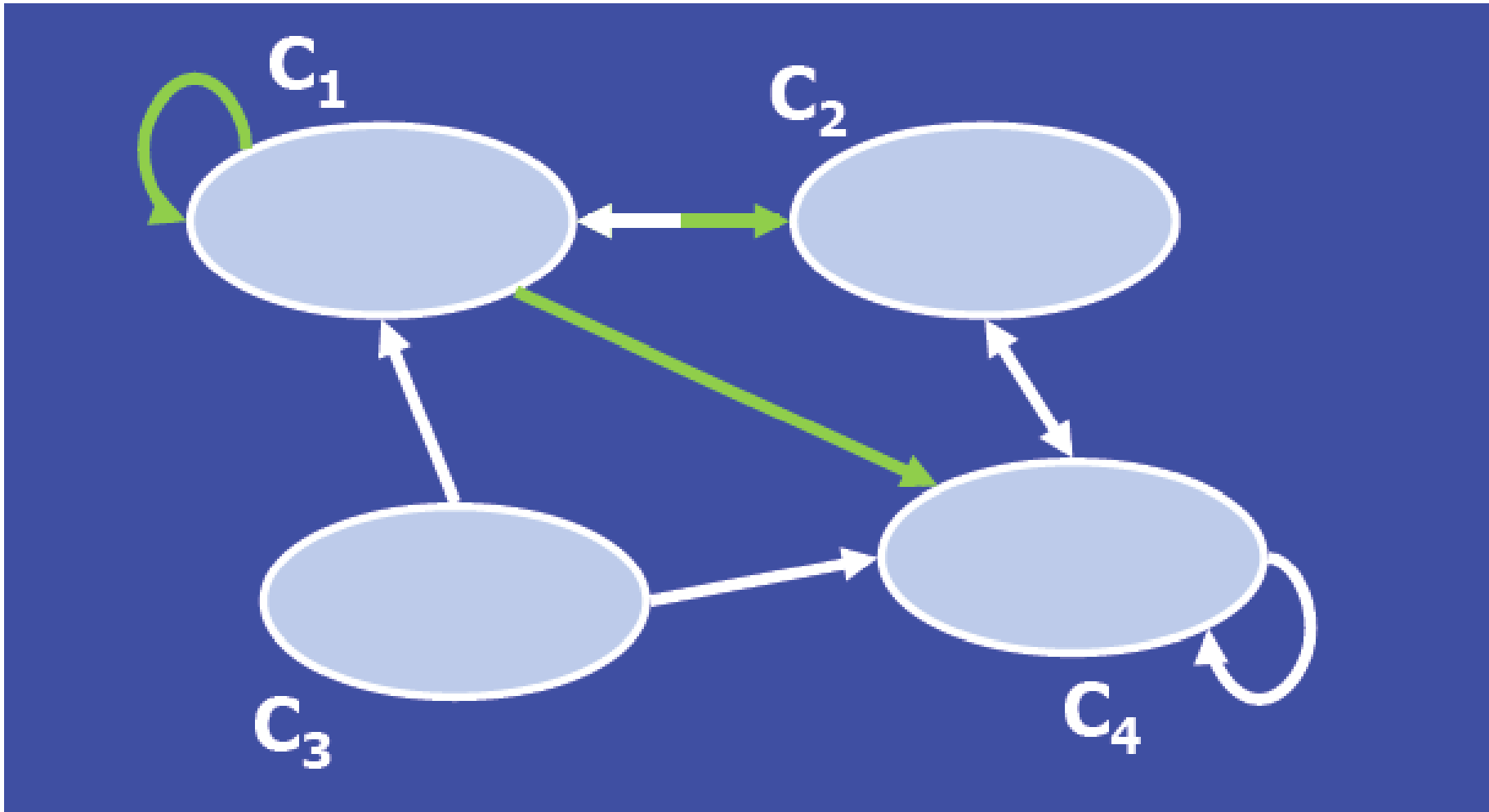


- For each car pairwise compare the criteria for importance to establish the priorities of the criteria in that car: Prestige, Price, MPG, and Comfort
- This results in a vector of priorities of the criteria for each car.

# Cars compared with respect to Prestige (and similarly for the other criteria)



- ii. Calculate priorities among groups



- ii. Calculate priorities among groups

**Pairwise comparison matrix**

$C_1$	$C_1$	$C_2$	$C_4$
$C_1$			
$C_2$			
$C_4$			

**QUESTION:** Given two groups of the system, C2 and C4, which have an influence on group C1, How much more does C2 influence on C1 than C4?

- ii. Calculate priorities among groups

The priorities vector has the following structure:

$$[w_{1,1}, w_{2,1}, w_{4,1}]^T$$

Before its introduction into the supermatrix this vector has to be completed with 0s:

$$[w_{1,1}, w_{2,1}, 0, w_{4,1}]^T$$

**SAME PROCEDURE FOR ALL THE CLUSTERS OF THE NETWORK.**

iii. Build original supermatrix

		$C_1$				$C_2$				...	$C_m$			
		$e_{11}$	$e_{12}$	...	$e_{1n_1}$	$e_{21}$	$e_{22}$	...	$e_{2n_2}$	...	$e_{m1}$	$e_{m2}$	...	$e_{mn_m}$
$C_1$	$e_{11}$	$W_{11}$				$W_{12}$				...	$W_{1m}$			
	$e_{12}$													
	...													
	$e_{1n_1}$													
$C_2$	$e_{21}$	$W_{21}$				$W_{22}$				...	$W_{2m}$			
	$e_{22}$													
	...													
	$e_{2n_2}$													
...	...	...				...	...							
$C_m$	$e_{m1}$	$W_{m1}$				$W_{m2}$				...	$W_{mm}$			
	$e_{m2}$													
	...													
	$e_{mn_m}$													

iii. Build original supermatrix

Influence matrix

		C <sub>1</sub>			C <sub>2</sub>			
		e <sub>11</sub>	e <sub>12</sub>	e <sub>13</sub>	e <sub>21</sub>	e <sub>22</sub>	e <sub>23</sub>	e <sub>24</sub>
C <sub>1</sub>	e <sub>11</sub>	0	1	0	1	1	0	0
	e <sub>12</sub>	1	0	1	1	0	0	1
	e <sub>13</sub>	1	1	0	1	1	0	1
C <sub>2</sub>	e <sub>21</sub>	1	0	1	0	0	0	0
	e <sub>22</sub>	0	1	1	0	0	0	0
	e <sub>23</sub>	1	0	1	0	0	0	0
	e <sub>24</sub>	1	1	1	0	0	0	0

Original supermatrix

		C <sub>1</sub>			C <sub>2</sub>			
		e <sub>11</sub>	e <sub>12</sub>	e <sub>13</sub>	e <sub>21</sub>	e <sub>22</sub>	e <sub>23</sub>	e <sub>24</sub>
C <sub>1</sub>	e <sub>11</sub>							
	e <sub>12</sub>							
	e <sub>13</sub>							
C <sub>2</sub>	e <sub>21</sub>	0,230	<b>W<sub>21,11</sub></b>					
	e <sub>22</sub>	0	<b>W<sub>22,11</sub></b>					
	e <sub>23</sub>	0,122	<b>W<sub>23,11</sub></b>					
	e <sub>24</sub>	0,648	<b>W<sub>24,11</sub></b>					

## ● Example: The Unweighted Supermatrix

For example, the priorities in the last column come from comparing the criteria with respect to the Honda Civic. That is, what we value most about the Honda is its price.

Cluster Node Labels		2Criteria				3Alternatives		
		1Prestige	2Price	3MPG	4Comfort	1Acura TL	2Toyota Camry	3Honda Civic
2Criteria	1Prestige	0.000000	0.000000	0.000000	0.000000	0.499407	0.275319	0.066315
	2Price	0.000000	0.000000	0.000000	0.000000	0.065910	0.148558	0.566142
	3MPG	0.000000	0.000000	0.000000	0.000000	0.079445	0.123967	0.247366
	4Comfort	0.000000	0.000000	0.000000	0.000000	0.355239	0.452155	0.120177
3Alternatives	1Acura TL	0.707117	0.063252	0.181821	0.704936	0.000000	0.000000	0.000000
	2Toyota Camry	0.070155	0.193881	0.272723	0.210920	0.000000	0.000000	0.000000
	3Honda Civic	0.222728	0.742867	0.545455	0.084144	0.000000	0.000000	0.000000

Done

● iv. Calculate weighted supermatrix

The original supermatrix has to become a matrix **stochastic by columns**. That way **it can converge** when raising it to successive powers.

For that the clusters priorities are needed.

iv. Calculate weighted supermatrix

### Weighted supermatrix

		$C_1$				$C_2$				...	$C_{m-1}$				$C_m$				
		$VE_{11}$	$VE_{12}$	...	$VE_{1m_1}$	$VE_{21}$	$VE_{22}$	...	$VE_{2m_2}$	...	$VE_{m-1,1}$	$VE_{m-1,2}$	...	$VE_{m-1,m-1}$	$A_1$	$A_2$	...	$A_{m-1}$	$A_p$
$C_1$	$VE_{11}$	$w_{1,1}$	$w_{1,1}$	$W_{11}$	$w_{1,2}$	$w_{1,2}$	$W_{12}$	...	$w_{1,m-1}$	$w_{1,m-1}$	$W_{1,m-1}$	$w_{1,m}$	$w_{1,m}$	$W_{1m}$					
	$VE_{12}$																		
	...																		
$VE_{1m_1}$																			
$C_2$	$VE_{21}$	$w_{2,1}$	$w_{2,1}$	$W_{21}$	$w_{2,2}$	$w_{2,2}$	$W_{22}$	...	$w_{2,m-1}$	$w_{2,m-1}$	$W_{2,m-1}$	$w_{2,m}$	$w_{2,m}$	$W_{2m}$					
	$VE_{22}$																		
	...																		
$VE_{2m_2}$																			
...	...	...	...	...	...	...	...	...	...	...	...	...	...						
$C_{m-1}$	$VE_{m-1,1}$	$w_{m-1,1}$	$w_{m-1,1}$	$W_{m-1,1}$	$w_{m-1,2}$	$w_{m-1,2}$	$W_{m-1,2}$	...	$w_{m-1,m-1}$	$w_{m-1,m-1}$	$W_{m-1,m-1}$	$w_{m-1,m}$	$w_{m-1,m}$	$W_{m-1,m}$					
	$VE_{m-1,2}$																		
	...																		
$VE_{m-1,m-1}$																			
$C_m$	$A_1$	$w_{m,1}$	$w_{m,1}$	$W_{m1}$	$w_{m,2}$	$w_{m,2}$	$W_{m2}$	...	$w_{m,m-1}$	$w_{m,m-1}$	$W_{m,m-1}$	$w_{m,m}$	$w_{m,m}$	$W_{mm}$					
	$A_2$																		
	...																		
	$A_{m-1}$																		
	$A_p$																		

● v. Calculate limit supermatrix

Raising the weighted supermatrix to limiting powers until all the entries converge. At the end, all the columns are the same and **their values show the global priority of each element of the network.**

v. Calculate limit supermatrix

From the limit supermatrix:

$C_{m-1}$	$W_{m-1, n_{m-1}}$			
	$A_1$			
$C_m$	$A_2$			
	...	...	...	...
	$A_{n_{m-1}}$			
	$A_p$			

Normalize the values of the alternatives and the criteria separately to obtain the priorities.

## Limit Supermatrix

The limit supermatrix is obtained by raising the weighted supermatrix to powers until it converges (in this case all columns are the same, though not always true).

Cluster Node Labels		2Criteria				3Alternatives		
		1Prestige	2Price	3MPG	4Comfort	1Acura TL	2Toyota Camry	3Honda Civic
2Criteria	1Prestige	0.150362	0.150362	0.150362	0.150362	0.150362	0.150362	0.150362
	2Price	0.132431	0.132431	0.132431	0.132431	0.132431	0.132431	0.132431
	3MPG	0.074569	0.074569	0.074569	0.074569	0.074569	0.074569	0.074569
	4Comfort	0.142637	0.142637	0.142637	0.142637	0.142637	0.142637	0.142637
3Alternatives	1Acura TL	0.228809	0.228809	0.228809	0.228809	0.228809	0.228809	0.228809
	2Toyota Camry	0.086646	0.086646	0.086646	0.086646	0.086646	0.086646	0.086646
	3Honda Civic	0.184545	0.184545	0.184545	0.184545	0.184545	0.184545	0.184545

The final answer, the synthesis, below is obtained by normalizing the raw values for the alternatives from the limit supermatrix:

Here are the overall synthesized priorities for the alternatives. You synthesized from the network Super Decisions Main Window: ANP2.mod

Name	Graphic	Ideals	Normals	Raw
1Acura TL	<div style="width: 100%;"></div>	1.000000	0.457617	0.228809
2Toyota Camry	<div style="width: 33%;"></div>	0.378685	0.173293	0.086646
3Honda Civic	<div style="width: 66%;"></div>	0.806547	0.369090	0.184545

	ANP	AHP
Acura TL	<b>.457</b>	.344
Toyota Camry	<b>.173</b>	.200
Honda Civic	<b>.369</b>	.455

## ● Why are the results different?

- **In AHP** – the user, going top-down making comparisons, when asked without referring to actual alternatives, over-estimated the importance of cost.
- **In ANP** – the user learned through feedback comparisons that his/her priority for Cost is not nearly as high as originally thought when asked the question abstractly, while Prestige gets more weight.

*This is why a husband and wife, setting out to the dealership to buy a “sensible” low priced car, walked out with something entirely different and much more costly. When they saw the shiny cars on the showroom floor they revised their priorities.*

## ● ANP vs. AHP:

### ■ Advantages

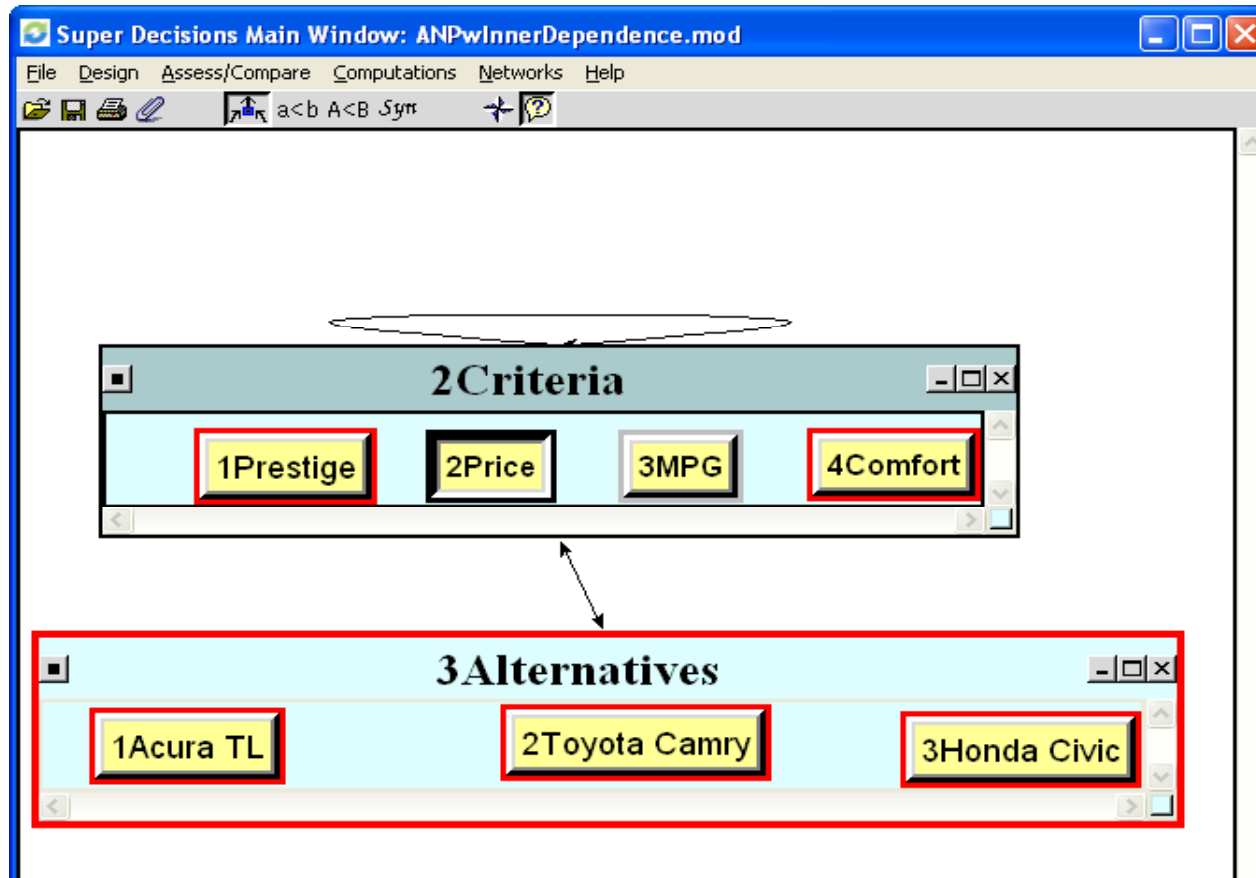
- ✓ More creative and flexible, it does not need a strict hierarchy to order the elements.
- ✓ It allows a closer model to reality.
- ✓ Results more precise and objective.

### ■ Disadvantages

- ✓ The higher complexity of the problem requires more calculations
- ✓ More time and effort are required



We can add some inner dependence: eg, price is influenced by prestige and comfort; and prestige is influenced by price and comfort.



## ● Inner Dependent Comparison

Which influences Price more, Prestige or Comfort?

Comparisons wrt "2Price" node in "2Criteria" cluster

File Computations Misc. Help

Graphic Verbal Matrix Questionnaire

Comparisons wrt "2Price" node in "2Criteria" cluster  
1Prestige is moderately more Preference than 4Comfort

1. 1Prestige    >=9.5 9 8 7 6 5 4 3 2 1 2 3 4 5 6 7 8 9    >=9.5 No comp. 4Comfort

Which influences Prestige more, Price or Comfort?

Comparisons wrt "1Prestige" node in "2Criteria" cluster

File Computations Misc. Help

Graphic Verbal Matrix Questionnaire

Comparisons wrt "1Prestige" node in "2Criteria" cluster  
2Price is strongly more Preference than 4Comfort

1. 2Price    >=9.5 9 8 7 6 5 4 3 2 1 2 3 4 5 6 7 8 9    >=9.5 No comp. 4Comfort